Matrix Algebra:

Analogue in $\mathbb{R}$
$1 \cdot a = a$ for all $a \in \mathbb{R}$

Identity matrix

: $I \in \mathcal{M}_{n,n}$ such that

$IA = A$ for all $A \in \mathcal{M}_{n,p}$

and $BI = B$ for all $B \in \mathcal{M}_{m,n}$

From direct computation, it is easy to see that

$$I = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix}$$

$a \cdot a^{-1} = a^{-1} \cdot a = 1$ for all $a \in \mathbb{R}, a \neq 0$

Matrix inversion

$A \in \mathcal{M}_{n,n}$, we say $B \in \mathcal{M}_{n,n}$ is the inverse of A, denoted by $B = A^{-1}$ if $BA = AB = I$.

When A^{-1} exists, we say A is nonsingular

Examples of singular matrix (A^{-1} does not exist)

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \left(\text{since } A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} \right)$$

Prop If A is nonsingular, then A^{-1} is unique.

pf: If $AB = BA = I$

and $AC = CA = I$

$\Rightarrow B = BI = B(AC) = (BA)C = IC = C$ ✱

Remark: We restrict the definition of A^{-1} only for $A \in \mathcal{M}_{n,n}$ (square matrix)

In fact, if $A \in \mathcal{M}_{m,n}$ with $m \neq n$, then A^{-1} does not exist.

Example: $A = \begin{pmatrix} a & b \end{pmatrix}$, $B = \begin{pmatrix} c \\ d \end{pmatrix}$

then $BA = \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} ca & cb \\ da & db \end{pmatrix}$

cannot be $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (Why?)

Matrix transpose

$$A \in \mathcal{M}_{m,n}, \quad A^T \overset{\text{def}}{=\!=} \begin{cases} A^T \in \mathcal{M}_{n,m} \\ (A^T)_{ij} = (A)_{ji} \end{cases}$$

Proposition:

$$(A^T)^T = A$$

$$(\alpha A)^T = \alpha A^T \qquad \alpha \in \mathbb{R}$$

$$(A+B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T$$

pf: for last one

$$(AB)^T_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki}$$

$$= \sum_k B_{ki} A_{jk}$$

$$= \sum_k (B^T)_{ik} (A^T)_{kj} = B^T A^T$$

Elementary Matrices

$A \in \mathcal{M}_{m,n}$ $\quad Ax = b$ is equivalent to

$$(MA)x = (Mb)$$

provided $M \in \mathcal{M}_{m,m}$ is invertible

$$\left(Ax = b \underset{M^{-1}\cdot}{\overset{M\cdot}{\Longleftrightarrow}} (MA)x = Mb \right)$$

The two systems can be derived from each other

Application: Find an invertible M such that $(MA)x = Mb$ is easier to solve

This is very similar to Gauss and Gauss-Jordan elemination.

In fact, each row operation in Gauss-Jordan multiplication corresponds to a matrix multiplication.

These matrix are called elementary matrices

Examples 1

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Effect of $E_1 A$: interchange first 2 rows of $A \in \mathcal{M}_{3,n}$

Effect of AE_1: interchange first 2 columns of $A \in \mathcal{M}_{m,3}$

Remark: Effect of E_1 applied to $I \in \mathcal{M}_{33}$

the Resulting matrix $= E_1$

Reason: $IE_1 = E_1 I = E_1$

Example 2

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Effect of $E_2 A$: (or what does $E_2 \cdot (\ )$ do to A?)

and Effect of $AE_2 = ?$

Example 3 What is the corresponding elementary matrix multiplication for the row operation $\begin{pmatrix} \text{---} \\ \text{---} \\ \text{---} \end{pmatrix} \times 5$?

Ans: Apply this operation to I to get E_3, the matrix multiplication is $E_3 \cdot (\ )$

Inverse of Elementary matrices

Fact: All row (column) operation of the 3 types of elementary matrices are reversible:

Row operation:	Reverse operations:
$\begin{pmatrix} \\ = \\ \\ \end{pmatrix}$ ith, jth	same
$\begin{pmatrix} \\ - \\ \\ \end{pmatrix}$ ith $\leftarrow \alpha \cdot$ $\alpha \neq 0$	$\begin{pmatrix} \\ \\ \\ \end{pmatrix}$ ith $\leftarrow \frac{1}{\alpha} \cdot$
$\begin{pmatrix} \\ \\ \\ \end{pmatrix}$ ith, jth $\downarrow \times m$	$\begin{pmatrix} \\ \\ \\ \end{pmatrix}$ ith, jth $\downarrow \times(-m)$

Corollary: All 3 types of elementary matrices are nonsingular there inverse matrices are elementary matrices of the same type.

1-18

Extra Homework for elementary matrices

This set of Homework is about "what does elementary matrices do to the matrix they multiply on"

(1) Give an example from each category of elementary 3×3 matrices,

(2) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

Compute E_iA and AE_i by

(a) direct calculation

(b) Figure out "what do $E_i\cdot(\,)$ and $(\,)\cdot E_i$ do to A" and apply them to A.

Hint: To find out what they do to A, just ask "How do you turn $I_{3,3}$ into E_i"?

(3) Can you write down

$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$ and $A \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

without direct Calculation?

Equivalent conditions of non-singularity

$A \in M_{n,n}$

(a) A is nonsingular

$\Leftrightarrow$ (b) $Ax=0$ implies $x=0$

(c) A is row equivalent to I

i.e. $\exists\ E_1, E_2 \ldots E_k$ elementary matrices such that

$\ldots\ I = E_k \cdots E_2 E_1 A$

Corollary: $A \in M_{n,n}$

$Ax = b$ has unique solution $\Leftrightarrow$ A is non-singular

Rm: (a) A is row equivalent to B

$\Leftrightarrow$ B " " " " A

(b) A is row equivalent to B

B is " " " " C

Then A " " " " C

1-20

A Deeper Understanding of matrix multiplication

$$\begin{pmatrix} & A & \end{pmatrix}\begin{pmatrix} \vec{x} \end{pmatrix} = \begin{pmatrix} \vec{y} \end{pmatrix}$$

$$A = \left(\begin{pmatrix} \vec{a}_1 \end{pmatrix} \begin{pmatrix} \vec{a}_2 \end{pmatrix} \cdots \begin{pmatrix} \vec{a}_n \end{pmatrix} \right)$$

Since $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

and $A\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \vec{a}_1, \cdots A\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \vec{a}_n$

$\therefore A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$

$$\Rightarrow \quad \vec{y} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

That is $\vec{y}$ is a "linear combination" of $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n$

Remark: This shows that, for $A\vec{x} = \vec{y}$ to have solutions, $\vec{y}$ must lie on the set of possible linear combinations of $\vec{a}_1, \cdots, \vec{a}_n$

That is $\vec{y} \in \{ x_1\vec{a}_1 + \cdots + x_n\vec{a}_n \mid x_1, x_2 \cdots x_n \in \mathbb{R} \}$

We will come back to this point at Chap 3

Example: (1) If $A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Then $A\vec{x} = \vec{y}$ has no solution

(2) If $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 4 \\ y_2 \end{pmatrix}$. Then

$A\vec{x} = \vec{y}$ has a solution(s) only if $y_2 = 8$.

Generalization to matrix • matrix

$$\underset{m}{(\quad A \quad)}\underset{n}{\left(\quad B \quad\right)} = \underset{m}{(\quad C \quad)}$$

$$(\quad A \quad)\left(\vec{b}_j\right) = \left(\vec{c}_j\right) \qquad j = 1, \ldots, p$$

$$\Rightarrow \vec{c}_j = b_{1j}\vec{a}_1 + b_{2j}\vec{a}_2 + \cdots + b_{nj}\vec{a}_n$$

Each ~~row~~ column of C is a "linear combination" of $\{\vec{a}_1, \ldots, \vec{a}_n\}$, ~~each~~ with different sets of coefficients $\{b_{1j}, \ldots b_{nj}\}$.

Alternative view: Rowwise

Block multiplication

Let $A = \begin{pmatrix} a(1,:) \\ a(2,:) \\ \vdots \\ a(m,:) \end{pmatrix}$ $B = \begin{pmatrix} b(:,1) & b(:,2) & \cdots & b(:,n) \end{pmatrix}$

where $a(i,:) = (a_{i1}, a_{i2}, \cdots a_{in})$

and $b(:,j) = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix}$

From direct computation, it is easy to see that

$$AB = \begin{pmatrix} Ab(:,1) & Ab(:,2) & \cdots & Ab(:,n) \end{pmatrix}$$

$$= \begin{pmatrix} a(1,:)B \\ a(2,:)B \\ \vdots \\ a(m,:)B \end{pmatrix}$$

Corollary: $A(B_1, B_2) = (AB_1 \;\; AB_2)$

where $A \in M_{m,n}$ $B_1 \in M_{n,p_1}$, $B_2 \in M_{n,p_2}$

Similarly $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} B = \begin{pmatrix} A_1 B \\ A_2 B \end{pmatrix}$

If we apply this repeatedly, we see that

if $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ $\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ =

$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$

Then $$AB = \begin{pmatrix} A\begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix} & A\begin{pmatrix} B_{12} \\ B_{22} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix} & \begin{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} A_{11} & A_{12} \end{pmatrix}\begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix} & \begin{pmatrix} A_{11} & A_{12} \end{pmatrix}\begin{pmatrix} B_{12} \\ B_{22} \end{pmatrix} \\ \begin{pmatrix} A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix} & \begin{pmatrix} A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{12} \\ B_{22} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^{2} A_{1k} B_{k1} & \sum_{k=1}^{2} A_{1k} B_{k2} \\ \sum_{k=1}^{2} A_{2k} B_{k1} & \sum_{k=1}^{2} A_{2k} B_{k2} \end{pmatrix}$$

$$= \text{Direct generalization of } \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Provided each A_{ik}, B_{kj} have correct size such that $A_{ik} B_{kj}$ is a valid multiplication

Homework: What are corresponding "elementary matrices" for block row/column operations?

Application of Block multiplication

Example: Solve

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$A_{11} \in M_{p_1 p_1}$ A_{11} invertible

$A_{22} \in M_{p_2 p_2}$

Ans:

Proceed with "Block version" of Gauss elemination

$$\left(\begin{array}{cc|c} A_{11} & A_{12} & b_1 \\ A_{21} & A_{22} & b_2 \end{array}\right) \xleftarrow{A_{11}^{-1}\cdot} \longrightarrow \left(\begin{array}{cc|c} I & A_{11}^{-1}A_{12} & A_{11}^{-1}b_1 \\ A_{21} & A_{22} & b_2 \end{array}\right) \downarrow -A_{21}\cdot$$

$$\rightarrow \left(\begin{array}{cc|c} I & A_{11}^{-1}A_{12} & A_{11}^{-1}b_1 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} & \end{array}\right)$$

$\Rightarrow$ necessary condition for non-singularity: $B = A_{22} - A_{21}A_{11}^{-1}A_{12}$ nonsingular

Special cases: (1) $A_{21} = 0$ or $A_{12} = 0$

B nonsingular $\Leftrightarrow$ A_{22} nonsingular

(2) $A_{22} \in M_{1,1}$

B nonsingular $\Leftrightarrow$ $A_{22} \neq A_{21}A_{11}^{-1}A_{12}$