Row echlon form : (Result of Gauss-elemination)

$$\begin{pmatrix} 1 & 0 & 2 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(i) First Nonzero entry in each row is 1

(ii) If k-th row is not ~~zero~~ identically zero

(number of leading zeros in (k+1)th row)

$>$ (k.)

(iii) If jth row is identically zero

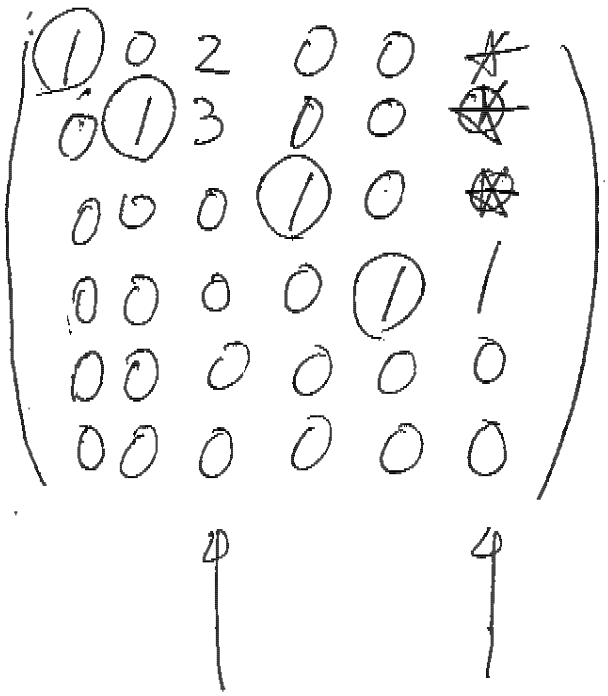
So are (j+1)th, (j+2)th, ----.

Reduced Row Echlon form.

(i) Row Echlon form

(ii) The first nonzero entry in each row is the only nonzero entry in its column.

Remark: The first nonzero element in each row appear exactly once in the resulting set of linear equations. (circled ones) they form a subset of all the variables. Since they appear exactly once in the final equation. We can simple choose them as the real "unknowns" and other variables as free variables to get all the solutions



(Result of Gauss-Jordan elemination)

Free variables.

1-7

$$\begin{pmatrix} 1 & 0 & 2 & 3 & 0 & 0 & \star \\ 0 & 1 & 4 & 6 & 0 & 0 & \star \\ 0 & 0 & 0 & 0 & 1 & 0 & \star \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Remark: A question has been raised in class regarding the choice of free variables and unknows.

The above example shows that other choice may lead to INCONSISTENCY!

HW: Show that the choice of

$\begin{cases} x_3, x_4, x_5, x_6 & \text{as unknowns} \\ x_1, x_2, x_7 & \text{as free variables} \end{cases}$

will lead to a inconsistent system.

Matrix Algebra

(i) indexing ~~order~~ rule

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{pmatrix}$$

(ii) Linear system of equation as Matrix • vector

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$$Ax = b$$

$$b_j = (a_{j1}, a_{j2} \cdots a_{jn}) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

= inner product on $\mathbb{R}^n$

(iii) Matrix multiplication as function (linear)

$$x \in \mathbb{R}^n \xrightarrow{\text{"A"}} b \in \mathbb{R}^m$$

$$b = Ax \qquad A \in \mathcal{M}_{m,n}$$

(iv) Matrix multiplication as composition of linear functions

$$x \in \mathbb{R}^n \xrightarrow{\text{"B"}} y \in \mathbb{R}^m \xrightarrow{\text{"A"}} b \in \mathbb{R}^p$$

$$y = Bx$$

$$b = Ay = \underline{\underline{AB}}x$$

$$(AB)_{ij} = ?$$

We want to define $C = AB$

so that $A(Bx) = Cx$

By manipulating the index expression, we can derive the desired formula

$$(AB)_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$$