Midterm

Dec 15, 1998 8:00-9:50 AM, closed book. No calculators allowed.

(1): (a): Write down the definition of $\lim_{x\to a} f(x) = L$, $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$ respectively.

(b): Prove that if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$, then $\lim_{x\to a} f(x) = L$.

(2): Find the tangent and normal line of $y = \sinh x$ at $(1, \sinh 1)$.

(3): Find the limits of the following expressions, distinguish between the cases when the limit is ∞ or $-\infty$ and the limit does not exist.

(a): $\lim_{x\to 0}(\ln \frac{1}{x})^x$

(b): $\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}$

(4): Find dy/dx for the following functions:

(a): $y = a^{a^x} + a^{x^a} + x^{a^a}$

(b): $y = \sin(\cos^2 x)\cos(\sin^2 x)$

(5): A cannon ball is to be fired toward the target located 1000 meters away and 100 m above the ground. The ball leaves the cannon with initial speed $v = 100m/s$. In order to hit the target, one has to adjust the angle of the cannon, θ, which is a unknown variable. The viscosity of the air will be ignored, therefore the horizontal component of the motion, $x(t)$, is constant-speeded while the vertical component, $y(t)$, is subject to a uniform gravity, $9.8m/s^2$ downward. For simplicity, we assume the cannon is located at origin, and the target at $(1000, 100)$.

(a): Write down $x(t)$ and $y(t)$ as functions of t, respectively.

(b): We would like to find an angle θ_0 such that the ball hits right on the target if fired at this angle. Write down Newton's formula to solve for θ_0. You are NOT required to get the numerical value of θ_0.

(6): Let f be a real valued function defined on $\{x \geq 0\}$ satisfying

(a): $f(0) = 1$,

(b): $f'(0) - 1$, and

(c): $f''(x) \leq 0$ for all $x \geq 0$.

Prove that $f(x) = 0$ has exactly one solution on $\{x \geq 0\}$.

(7): Graph $y = xe^{-x^2/2}$ over the interval $|x| \leq 3$, indicate critical points and inflection points, and determine whether the critical points are local extremums.

(8): $f(x) = e^{\frac{-1}{x^2}}$ for $x \neq 0$ and $f(0)$ is not defined.

(a): Show that f has a removable singularity at 0.

(b): With the singularity removed, f is now a function defined for all real x. Show that f is differentiable at 0. What is $f'(0)$?

(9): Let f be a twice differentiable function on (a, b) with $f''(\xi) > 0$ for some ξ, $a < \xi < b$. Show that there exist x_1, x_2 in (a, b) such that $\frac{f(x_1)-f(x_2)}{x_1-x_2} = f'(\xi)$. (Hint: What can you say about $F(x) = f(x) - f'(\xi)x$?)