Study Guide and some remarks for Chap 02

1. Find and memorize typical examples in which the limit does not exists, not continuous, not differentiable, etc.
2. Study Sandwich Theorem and applications. Practice on variants of $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$.
3. Study the precise definitions on p77, p84, p87, p93, p104, p110 and p116. Be able to write down these definitions and find an example for each definition and verify with the $\varepsilon-\delta$ argument.
4. Study how to prove $\lim_{x\to c} f(x) = L$ using standard tricks such as the $\epsilon/2$ argument. Study how to disprove $\lim_{x\to c} f(x) = L$.
5. Study how the $\varepsilon-\delta$ argument can be used to examine statements like Example 6, 7 in section 2.3 and Theorem 10 in section 2.5.
6. Study Intermediate Value Theorem and its application.

Remark 1:

The standard form of formal definition of limit is

$$\begin{aligned} &\lim_{x\to c} f(x) = L \\ \iff\ &\text{For any } \varepsilon > 0, \\ &\ \text{there exists a } \delta > 0 \text{ such that, } 0 < |x-c| < \delta \implies |f(x)-L| < \varepsilon. \qquad (1) \end{aligned}$$

In practice, however, we only need to show that (1) holds for small enough $\varepsilon > 0$. This is convenient in many examples. See for instance, the square root in Example 5 of section 2.3.

In other words, we can use

$$\begin{aligned} &\lim_{x\to c} f(x) = L \\ \iff\ &\text{For any } \varepsilon \in (0,\varepsilon_0),\ \varepsilon_0 > 0, \\ &\ \text{there exists a } \delta > 0 \text{ such that, } 0 < |x-c| < \delta \implies |f(x)-L| < \varepsilon. \end{aligned}$$

The reason is, suppose that (1) holds for all $\varepsilon \in (0,\varepsilon_0)$, where $\varepsilon_0 > 0$. Then in particular, it holds for $\varepsilon = \varepsilon_0/2$ and therefore we can find the corresponding δ in (1). Denote this particular δ by δ_0, then the statement (1) reads,

$$0 < |x-c| < \delta_0 \implies |f(x)-L| < \varepsilon_0/2. \qquad (2)$$

This implies that (1) not only holds for $\varepsilon \in (0,\varepsilon_0)$, it actually holds for any $\varepsilon > 0$. To see this, we simply take $\delta = \delta_0$ for those $\varepsilon \geq \varepsilon_0$. Then from (2), we have

$$0 < |x-c| < \delta_0 \implies |f(x)-L| < \varepsilon_0/2 < \varepsilon. \qquad (3)$$

This shows that taking $\delta = \delta_0$ works for those $\varepsilon \geq \varepsilon_0$.

Remark 2:

Formal definition of '$\lim_{x\to c} f(x) \neq L$', or '$\lim_{x\to c} f(x) = L$ is false':

We know that

$$\begin{aligned} & \lim_{x\to c} f(x) = L \quad \text{is true} \\ \iff & \text{For any } \varepsilon > 0,\ \text{there exists a } \delta > 0 \text{ such that,} \\ & 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon. \end{aligned} \tag{4}$$

therefore

$$\begin{aligned} & \lim_{x\to c} f(x) = L \quad \text{is false} \\ \iff & \text{There exists an } \varepsilon > 0 \text{ such that,} \\ & \text{for any } \delta > 0,\ \text{the statement (4) is false.} \end{aligned}$$

On the other hand,

$$\begin{aligned} & \text{the statement (4) is false} \\ \iff & 0 < |x - c| < \delta \;\not\Longrightarrow\; |f(x) - L| < \varepsilon \\ \iff & \text{there is an } x \text{ satisfying } 0 < |x - c| < \delta \text{ and } |f(x) - L| \geq \varepsilon \end{aligned}$$

combining these statements together, we conclude that

$$\begin{aligned} & \lim_{x\to c} f(x) = L \quad \text{is false} \\ \iff & \text{There exists an } \varepsilon > 0 \text{ such that,} && (5) \\ & \text{for any } \delta > 0,\ \text{there is an } x \text{ satisfying } 0 < |x - c| < \delta \text{ and } |f(x) - L| \geq \varepsilon && (6) \end{aligned}$$

In order to prove $\lim_{x\to c} f(x) = L$ is false, we first need to find a correct ε in (5) (usually by inspecting the graph of $y = f(x)$ near $x = c$), then describe how to find x from δ in (6).