Midterm 2

May 10, 2016

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1. **(10 pts)** True or False?

   If $f(x,y)$ is differentiable at $(0,0)$, then $f(x,y)$ is continuous at $(0,0)$.
   Answer. True. **(2 pts)**
   Since f is differentiable at $(0,0)$, both $f_x(0,0)$ and $f_y(0,0)$ exist **(2 pts)** and

   $f(x,y)-f(0,0) = f_x(0,0)x+f_y(0,0)y+\epsilon_1 x+\epsilon_2 y$ where $\epsilon_1, \epsilon_2 \to 0$ as $x, y \to 0$ **(4 pts)**

   $\Rightarrow |f(x,y)-f(0,0)| = |f_x(0,0)||x|+|f_y(0,0)||y|+|\epsilon_1||x|+|\epsilon_2||y| \to 0$ as $(x,y) \to (0,0)$. (**2pts**)

2. **(10 pts)** Evaluate

   $$\frac{d}{dy}\int_1^{2+y^2} \frac{\cos(xy)}{x}dx.$$

   Answer. Let $G(y,z) = \int_1^z \frac{\cos(xy)}{x}\,dx$. **(2 pts)** Then we need to compute

   $$(G(y,2+y^2))' = G_y(y,2+y^2)+G_z(y,2+y^2)\cdot 2y = -\int_1^{2+y^2} \sin(xy)\,dx+\frac{\cos((2+y^2)y)}{2+y^2}\cdot 2y \text{ (4 pts)}$$

   $$= \frac{1}{y}[\cos((2+y^2)y) - \cos y] + \frac{\cos((2+y^2)y)}{2+y^2}\cdot 2y. \text{ (4pts)}$$

3. **(10 pts)** Find the equation of plane normal to the following curve at $(1,-1,1)$

   $$\begin{cases} x^2+2y^2+3z^2 = 6 \\ x+y+z = 1 \end{cases}$$

   Answer. First compute the two gradients at $(1,-1,1)$

   $$(2x,4y,6z)_{(1,-1,1)} = (2,-4,6) \text{ (2 pts)}$$

   and

   $$(1,1,1)|_{(1,-1,1)} = (1,1,1). \text{ (2 pts)}$$

   The normal vector of the plane is parallel to the outer product of these two gradients:

   $$n = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = (5,-2,-3). \text{ (2 pts)}$$

   Therefore, the equation of the plane is

   $$5(x-1)-2(y+1)-3(z-1) = 0. \text{ (4 pts)}$$

4. **(10 pts)**Show that, for all $a \in \mathbb{R}$, the point $(x_0, y_0) = (0,0)$ is a critical point of the function

$$f_a(x,y) = (a-1)(x+y)^2 + (a+1)(x-2y)^2.$$

For what values of a is the point $(0,0)$ a local minimum, a local maximum and a saddle point, respectively?

Answer.

Method.1 For $a \geq 1$, $f_a(x,y) = (a-1)(x+y)^2 + (a+1)(x-2y)^2$ is a sum of two squares. And $(0,0)$ admits the minimum 0. **(2 pts)**
For $-1 < a < 1$, $f_a(x,y) = (a+1)(x-2y)^2 - (1-a)(x+y)^2$ is a difference of two squares. And $(0,0)$ is a saddle point since $f_a(1,-1) = 9(a+1) > 0$ and $f_a(2,1) = -9(1-a) < 0$. **(6 pts)**
For $a \leq -1$, $f_a(x,y) = -[(1-a)(x+y)^2 + (-1-a)(x-2y)^2]$ is a negative sum of two squares. And $(0,0)$ admits the maximum 0. **(2 pts)**

Method.2 First compute the gradient at $(0,0)$

$$\nabla f_a(0,0) = (2(a-1)(x+y)+2(a+1)(x-2y),\ 2(a-1)(x+y)+2(a+1)(x-2y)(-2))|_{(0,0)} = (0,0),$$

and thus $(0,0)$ is a critical point. **(2 pts)**

Now compute the second derivatives **(2 pts)**

$$(f_a)_{xx}(0,0) = 2(a-1) + 2(a+1) = 4a,$$

$$(f_a)_{yy}(0,0) = 2(a-1) + 8(a+1) = 10a + 6 = 2(5a+3),$$

and

$$(f_a)_{xy}(0,0) = 2(a-1) - 4(a+1) = -2(a+3).$$

A local minimum occurs as $(f_a)_{xx} > 0$ and $(f_a)_{xx}(f_a)_{yy} - (f_a)_{xy}^2 > 0$, i.e.

$$\begin{aligned} & a > 0,\ 8a(5a+3) - 4(a+3)^2 = 36(a^2-1) > 0 \\ \Leftrightarrow \quad & a > 1. \ \textbf{(2 pts)} \end{aligned}$$

A local maximum occurs as $(f_a)_{xx} < 0$ and $(f_a)_{xx}(f_a)_{yy} - (f_a)_{xy}^2 > 0$, i.e.

$$\begin{aligned} & a > 0,\ 8a(5a+3) - 4(a+3)^2 = 36(a^2-1) > 0 \\ \Leftrightarrow \quad & a < -1. \ \textbf{(2 pts)} \end{aligned}$$

A saddle point occurs as $(f_a)_{xx}(f_a)_{yy} - (f_a)_{xy}^2 < 0$, i.e.

$$\begin{aligned} & 8a(5a+3) - 4(a+3)^2 = 36(a^2-1) < 0 \\ \Leftrightarrow \quad & -1 < a < 1. \ \textbf{(2 pts)} \end{aligned}$$

For $a = 1$, $f_a(x,y) = 2(x-2y)^2$. And thus $(0,0)$ admits a local minimum.
For $a = -1$, $f_a(x,y) = -2(x+y)^2$. And this $(0,0)$ admits a local maximum.

Method.1

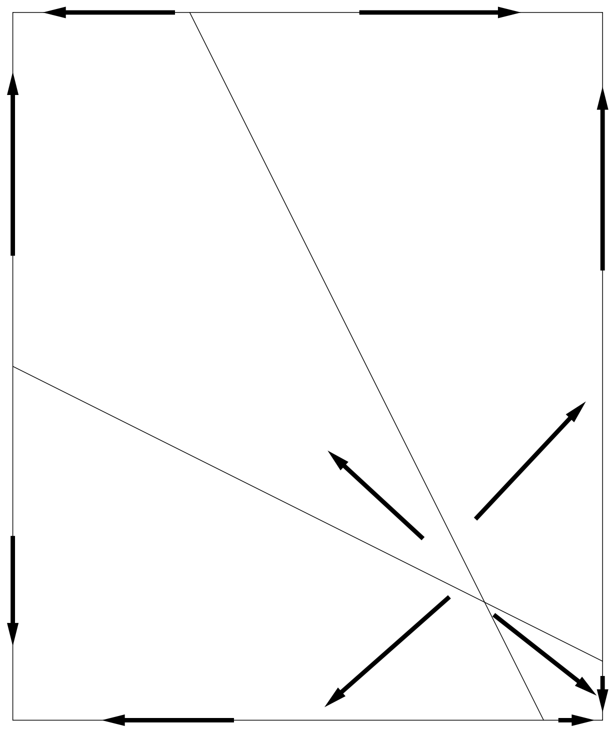

Figure 1: Method 1 for problem 5, The gradient analysis.

Getting the two equalities $a = \pm 1$ correctly: **(extra 2 pts)**.

5. **(10 pts)**Find absolute maxima and minima of $f(x, y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular $0 \le x \le 5$, $-3 \le y \le 3$.
Answer.

First find the gradient:

$$\nabla f(x, y) = (2x + y - 6, x + 2y). \textbf{ (2 pts)}$$

Therefore one can plot the gradients as in figure 1. From the plot it is easy to see that $f(4, -2) = -10$ is indeed local and absolute minima. Moreover, the local maximum consists of the four corners. Upon comparing the values of f on the four corners, it follows that the absolute maxima is $f(5, 3) = 21$.

Method.2 First find the interior critical points:

$$\nabla f(x, y) = (2x+y-6, x+2y) = (0, 0) \Rightarrow (x, y) = (4, -2), \ f(4, -2) = -10. \textbf{ (2 pts)}$$

For $x = 0$, $f(0, y) = y^2 + 2$, $-3 \le y \le 3$. Compute $(f(0, y))' = 2y = 0 \Rightarrow y = 0$, and thus we compare these values

$$f(0, 0) = 2, \ f(0, -3) = f(0, 3) = 11. \textbf{ (1 pt)}$$

For $x = 5$, $f(5, y) = y^2 + 5y - 3$, $-3 \le y \le 3$. Compute $(f(5, y))' = 2y + 5 = 0 \Rightarrow y = -5/2$, and thus we compare these values

$$f(5, -3) = -9,\ f(5, 3) = 21,\ f(5, -5/2) = -\frac{37}{4}.\ \textbf{(1 pt)}$$

For $y = -3$, $f(x, -3) = x^2 - 9x + 11$, $0 \le x \le 5$. Compute $(f(x, -3))' = 2x - 9 = 0 \Rightarrow x = 9/2$, and thus we compare these values

$$f(0, -3) = 11,\ f(9/2, -3) = -\frac{37}{4},\ f(5, -3) = -9.\ \textbf{(1 pt)}$$

For $y = 3$, $f(x, 3) = x^2 - 3x + 11$, $0 \le x \le 5$. Compute $(f(x, 3))' = 2x - 3 = 0 \Rightarrow x = 3/2$, and thus we compare these values

$$f(0, 3) = 11,\ f(3/2, 3) = \frac{35}{4},\ f(5, 3) = 21.\ \textbf{(1 pt)}$$

Therefore, the absolute maxima is $f(5, 3) = 21$ and the absolute minimum is $f(4, -2) = -10$. **(4 pts)**

6. **(10 pts)**Give an example of a constraint optimization problem that, upon applying the method of Lagrangian multipliers, results in a system of 5 equations with 5 unknowns (and write down the equations). Need not solve it.
Answer. Find the points on the intersection curve of these two surfaces $x^2+2y^2+3z^2 = 1$ and $z = 0$ which admit the shortest distance from the origin. Let $f(x, y, z) = x^2 + y^2 + z^2$, $g_1(x, y, z) = x^2 + 2y^2 + 3z^2 - 1$, and $g_2(x, y, z) = z$. **(5 pts)** Then applying the method of Lagrangian multipliers result in this system

$$\begin{cases} 2x = 2\lambda_1 x \\ 2y = 4\lambda_1 y \\ 2z = 6\lambda_1 z + \lambda_2 \\ x^2 + 2y^2 + 3z^2 - 1 = 0 \\ z = 0.\ \textbf{(5 pts)} \end{cases}$$

7. **(10 pts)**Use Taylor's formula to find the quadratic approximation of $f(x, y, z) = \dfrac{1}{1 - x - y + z}$ near the origin.
Answer. Find all of the first and second derivatives first.

$$f_x(0, 0, 0) = f_y(0, 0, 0) = -f_z(0, 0, 0) = \left.\frac{1}{(1 - x - y + z)^2}\right|_{(0,0,0)} = 1,\ \textbf{(2 pts)}$$

and

$$f_{xx}(0, 0, 0) = f_{yy}(0, 0, 0) = f_{xy}(0, 0, 0) = -f_{zz}(0, 0, 0) = -f_{yz}(0, 0, 0)$$
$$= -f_{zx}(0, 0, 0) = \frac{2}{(1 - x - y + z)^3} = 2.\ \textbf{(2 pts)}$$

Therefore, the gradratic approximation is

$$\begin{aligned}
Q(x,y,z) &= f(0,0,0) + f_x(0,0,0)x + f_y(0,0,0)y + f_z(0,0,0)z \\
&\quad + \frac{1}{2}\Big[f_{xx}(0,0,0)x^2 + f_{yy}(0,0,0)y^2 + f_{zz}(0,0,0)z^2 \\
&\quad + 2f_{xy}(0,0,0)xy + 2f_{yz}(0,0,0)yz + 2f_{zx}(0,0,0)zx \Big] \textbf{ (4 pts)} \\
&= 1 + x + y - z + x^2 + y^2 - z^2 + 2xy - 2yz - 2zx. \textbf{ (2 pts)}
\end{aligned}$$

8. **(10 pts)** Evaluate $\left(\frac{\partial u}{\partial x}\right)_y$ at $(x,y,z,w) = (1,1,1,1)$ where $u(x,y,z,w) = x^2+y^2+z^2+w^2$ with the constraint $x+y+z+w=4$ and $x-y+z-w=0$.
Answer. First we compute z_x and w_z at $(1,1,1,1)$.

$$\begin{aligned}
& 1 + z_x + w_x = 0 \textbf{ (2 pts)} \\
& 1 + z_x - w_x = 0 \textbf{ (2 pts)} \\
\Rightarrow \quad & z_x = -1, w_x = 0. \textbf{ (2 pts)}
\end{aligned}$$

Now we compute $\left(\frac{\partial u}{\partial x}\right)_y$ at $(1,1,1,1)$.

$$\left(\frac{\partial u}{\partial x}\right)_y = 2x + 2zz_x + 2ww_x|_{(1,1,1,1)} \textbf{ (2 pts)} = 2 - 2 = 0. \textbf{ (2 pts)}$$

9. **(10 pts)** Evaluate

$$\int_0^2 \int_y^2 x^2 \cos(xy)\, dxdy$$

Answer.

$$\begin{aligned}
\int_0^2 \int_y^2 x^2 \cos(xy)\, dxdy &= \int_0^2 \int_0^x x^2 \cos(xy)\, dydx \textbf{ (4 pts)} \\
&= \int_0^2 x \sin(xy)\Big|_0^x \, dx \\
&= \int_0^2 x \sin(x^2)\, dx \textbf{ (3 pts)} \\
&= -\frac{1}{2}\cos(x^2)\Big|_0^2 = -\frac{1}{2}(\cos 4 - 1). \textbf{ (3 pts)}
\end{aligned}$$

10. **(10 pts)** Change

$$\int_{\sqrt{2}}^2 \int_{\sqrt{4-y^2}}^y dxdy$$

into an equivalent polar integral and evaluate the polar integral.
Answer.

$$\begin{aligned}
\int_{\sqrt{2}}^{2}\int_{\sqrt{4-y^2}}^{y} dxdy &= \int_{\pi/4}^{\pi/2}\int_{2}^{2\csc\theta} r\ drd\theta \textbf{ (2 pts)} \\
&= \int_{\pi/4}^{\pi/2} \left.\frac{r^2}{2}\right|_{2}^{2\csc\theta} d\theta \textbf{ (2 pts)} \\
&= \int_{\pi/4}^{\pi/2} (2\csc^2\theta - 2)\, d\theta \textbf{ (2 pts)} \\
&= (-2\cot\theta - 2\theta)\Big|_{\pi/4}^{\pi/2} \textbf{ (2 pts)} = 2 - \frac{\pi}{2}. \textbf{ (2 pts)}
\end{aligned}$$

11. **(10 pts)** Is $\int_1^\infty \sin\frac{1}{x^2}$ convergent? Explain.
Answer. Yes. **(2 pts)**

$$\because \lim_{x\to\infty}\frac{\sin\frac{1}{x^2}}{\frac{1}{x^2}} = 1 \in (0,\infty) \textbf{ (4 pts)} \text{ and } \int_1^\infty \frac{1}{x^2}\,dx \text{ converges. } \textbf{(4 pts)}$$

$\therefore$ we can conclude from Limit Comparison Test that the improper integral converges.

12. **(10 pts)** Evaluate $\sum_{n=0}^{\infty}\frac{x^n}{n+2}$ on $|x|<1$ using computational rules of power series.
Answer. The value is 1/2 as $x = 0$. **(2 pts)** Consider $x \neq 0$.

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{x^n}{n+2} &= x^{-2}\left(\sum_{n=0}^{\infty}\frac{x^{n+2}}{n+2}\right) \\
&= x^{-2}\left(\sum_{n=0}^{\infty}\int_0^x t^{n+1}\,dt\right) \textbf{ (2 pts)} \\
&= x^{-2}\left(\int_0^x \sum_{n=0}^{\infty} t^{n+1}\,dt\right) \\
&= x^{-2}\left(\int_0^x \frac{t}{1-t}\,dt\right) \textbf{ (2 pts)} \\
&= x^{-2}\left(\int_0^x -1+\frac{1}{1-t}\,dt\right) \\
&= x^{-2}\left(-x-\ln(1-x)\right) \textbf{ (2 pts)} \\
&= -\frac{1}{x} - \frac{\ln(1-x)}{x^2}. \textbf{ (2 pts)}
\end{aligned}$$