Midterm 1

Mar 28, 2016

Show all details.

1. (16 pts) Are the integrals $\int_0^1 \frac{1}{\sqrt{x+x^3}}dx$ and $\int_1^\infty \tan(\frac{1}{x})dx$ convergent? Explain.

Answer.

$$\because \lim_{x\to 0} \frac{\frac{1}{\sqrt{x+x^3}}}{\frac{1}{\sqrt{x}}} = 1 \textbf{ (3 pts)} \text{ and } \int_0^1 \frac{1}{\sqrt{x}}\,dx \text{ converges. } \textbf{(3 pts)} \therefore \text{ Converges. } \textbf{(2 pts)}$$

$$\because \lim_{x\to\infty} \frac{\tan(1/x)}{1/x} = 1 \textbf{ (3 pts)} \text{ and } \int_1^\infty \frac{1}{x}\,dx \text{ diverges. } \textbf{(3 pts)} \therefore \text{ Diverges. } \textbf{(2 pts)}$$

2. (6 pts) Give formal definition of $\lim_{n\to\infty} a_n = L$.

Answer.

$$\forall\, \epsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that } |a_n - L| < \epsilon\ \forall\, n \geq N. \textbf{ (6 pts)}$$

3. (8 pts) Evaluate $\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^n$.

Answer.

$$\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^n = \lim_{n\to\infty}\left(1-\frac{2}{n+1}\right)^{n+1}\frac{1}{1-\frac{2}{n+1}} \textbf{ (6 pts)} = e^{-2}. \textbf{ (2 pts)}$$

4. (6+10 pts) Show that $\sum_{k=1}^{\infty} k^{-2}$ converges and evaluate* $\lim_{n\to\infty}\frac{\log\left(\sum_{k=n}^\infty k^{-2}\right)}{\log n}$. Give details.

(If the limit is p, this means that $\sum_{k=n}^\infty k^{-2}$ is approximately n^p. Find p and prove it.)

Hint: recall the proof of one of the convergence tests.

Answer.

$$\because\ f(k) = k^{-2}\ \geq 0 \searrow \text{ ctu } \textbf{(2 pts)} \text{ and } \int_1^\infty \frac{1}{x^2}\,dx \text{ converges. } \textbf{(2 pts)} \therefore \text{ Converges. } \textbf{(2 pts)}$$

$\because$

$$\frac{1}{n} = \int_n^\infty \frac{1}{x^2}\,dx \textbf{ (2 pts)} \leq \sum_{k=n}^\infty \frac{1}{k^2} \leq \frac{1}{n^2} + \int_{n+1}^\infty \frac{1}{x^2}\,dx = \frac{1}{n^2} + \frac{1}{n+1} \leq \frac{2}{n+1} \textbf{ (2 pts)}$$

$$\Rightarrow -1 \textbf{ (2 pts)} \leq \frac{\log\left(\sum_{k=n}^\infty k^{-2}\right)}{\log n} \leq \frac{\log 2}{\log n} - \frac{\log(n+1)}{\log n} \textbf{ (2 pts)} \to -1.$$

$\therefore\ p = -1.$ **(2 pts)**

5. (8 pts) State (need not prove) Taylor's Theorem (or Taylor's formula). Assume the function f has derivatives of all orders on $\mathbb{R}$. Write down n terms approximation and the formula of the remainder term.
Answer. If f has derivatives of all orders in an open interval I containing a, then $\forall\, n \in \mathbb{N}$, $\forall\, x \in I$, $\exists\, c$ between a and x such that

$$f(x) = f(a)+f'(a)(x-a)+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n \ \textbf{(4 pts)}\ +\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.\ \textbf{(4 pts)}$$

6. (6+12 pts)

   (a) Show that the series $1 - \frac{1}{2\cdot 1!} + \frac{1}{4\cdot 2!} - \cdots + (-1)^n \frac{1}{2^n\cdot n!} + \cdots$ converges *absolutely*.

   (b) Find the sum of the series in (a). Prove your answer (that is, the equality holds).

Answer.

(a)

$$\lim_{n\to\infty} \frac{\frac{1}{2^{n+1}(n+1)!}}{\frac{1}{2^n n!}} \to 0.\ \textbf{(6 pts)}$$

(b)

$$e^{-1/2}.\ \textbf{(2 pts)}\ \ |R_n(-\frac{1}{2})| \le \frac{e^{-c}}{(n+1)!}\frac{1}{2^{n+1}} \to 0 \text{ where } c \in (0, \frac{1}{2})\ .\ \textbf{(10 pts)}$$

7. (8 pts) Find $\sum_{n=1}^{\infty} nx^n$ and $\sum_{n=1}^{\infty} n^2x^n$ on $|x| < 1$ using computational rules of power series. Need NOT prove your equality holds as in previous question.
Answer.

$$\begin{aligned}
& 1+x+x^2+\ldots = \frac{1}{1-x}\ \textbf{(2 pts)}\\
\Rightarrow\quad & x+2x^2+3x^3+\ldots = x\left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2}\ \textbf{(3 pts)}\\
\Rightarrow\quad & x+2^2x^2+3^2x^3+\ldots = x\left(\frac{x}{(1-x)^2}\right)' = \frac{x(1+x)}{(1-x)^3}\ \textbf{(3 pts)}.
\end{aligned}$$

8. (6 pts) Find a power series that converges on $[1, 5)$ and diverges elsewhere. Explain.
Answer.

$$\sum_n \frac{1}{n}\left(\frac{x-3}{2}\right)^n.\ \textbf{(2 pts)}$$

$$\lim_{n\to\infty} \frac{n}{n+1}\frac{1}{2} = \frac{1}{2} \Rightarrow R = 2.\ \textbf{(2 pts)}$$

For $x = 1$,

$$\sum_n \frac{(-1)^n}{n} \text{ converges. } \textbf{(1 pt)}$$

For $x = 5$,

$$\sum_n \frac{1}{n} \text{ diverges. } \mathbf{(1\ pt)}$$

9. (12 pts) Give an approximation of $\int_0^{\frac{1}{2}} \sin(x^2)dx$ to within 10^{-8}. Give the formula of the approximation, but need not find the numerical value. Explain why the error is less than 10^{-8}.
Answer.

$$\left| \int_0^{\frac{1}{2}} (-1)^n \frac{x^{2(2n+1)}}{(2n+1)!} dx \right| \mathbf{(4\ pts)} \leq \frac{1}{(4n+3) \cdot 2^{4n+3} \cdot (2n+1)!} < 10^{-8}\ \mathbf{(4\ pts)}.$$

$n \geq 3$ will do **(2 pts)**. The approximation is

$$\sum_{k=0}^{2} (-1)^k \frac{1}{(4k+3) \cdot 2^{4k+3} \cdot (2k+1)!}\ \mathbf{(2\ pts)}.$$

$(error \approx 4 \times 10^{-10})$
Giving optimal n, $(n = 3)$: **(extra 2 pts)**.

10. (8+10+10 pts) True or False? Prove it if true, give a counter example if false.

(a) If $\sum a_n$ converges, then $\sum na_n$ converges.

(b) * If $\sum a_n x^n$ converges on $|x| < 1$, then $\sum \sqrt{n} a_n x^n$ also converges on $|x| < 1$.

(c) If $g(x) = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ converges on $|x| < 1$, then $f(x) = g(x)$ on $|x| < 1$.

Answer.

(a) False **(2 pts)**. $\sum_n \frac{(-1)^n}{n}$ converges **(3 pts)**, but $\sum_n (-1)^n$ diverges **(3 pts)**.

(b) True **(2 pts)**. Given $|x| < 1$. Take y with $|x| < |y| < 1$. Then $|\sqrt{n} a_n x^n| \leq |a_n y^n|$ for n large enough since $\sqrt{n}|x/y|^n \to 0$ as $n \to \infty$ **(6 pts)**. Furthermore, $\sum_n |a_n y^n|$ converges **(2 pts)**.

(c) False **(2 pts)**. Take $f(x) = e^{-1/x^2}$ for $x \neq 0$, and $f(x) = 0$ for $x = 0$ **(3 pts)**. Then $f^{(n)}(0) = 0$, $\forall\, n \geq 0$ **(2 pts)**. Therefore $f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 \neq f(x)$, if $x \neq 0$ **(3 pts)**.