Brief answers to Final Exam

Jun 16, 2016

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1. (10 pts) Replace

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{2-r^2}} rdz\,dr\,d\theta$$

by triple integrals in spherical coordinates in the order $d\rho d\phi d\theta$ and $d\phi d\rho d\theta$, respectively. Need not evaluate them.

Answer.

$$\int_0^{2\pi}\left[\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}}\rho^2\sin\phi\,d\rho d\phi+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\csc\phi}\rho^2\sin\phi\,d\rho d\phi\right]d\theta\ \textbf{(5 pts)}$$

$$\int_0^{2\pi}\left[\int_0^{1}\int_0^{\frac{\pi}{2}}\rho^2\sin\phi\,d\phi d\rho+\int_1^{\sqrt{2}}\int_0^{\csc^{-1}\rho}\rho^2\sin\phi\,d\phi d\rho\right]d\theta\ \textbf{(5 pts)}$$

2. (15 pts) Evaluate the surface area of $S=\{z=\sqrt{x^2+y^2},\ 1\le xy\le 2,\ 1\le x/y\le 3,\ x>0,\ y>0\}$.

Answer.

Method.1 Let $f(x,y)=\sqrt{x^2+y^2}$. First compute

$$f_x=\frac{x}{\sqrt{x^2+y^2}},\ f_y=\frac{y}{\sqrt{x^2+y^2}}.\ \textbf{(2 pts)}$$

Then the surface area is

$$Area=\iint_R\sqrt{f_x^2+f_y^2+1}\,dxdy\ (\textbf{2pts})=\iint_R\sqrt{2}\,dxdy.\ (\textbf{1pt})$$

Let $u=xy$, $v=\frac{x}{y}$, $x,y>0$ **(2 pts)**. Then $1\le u\le 2$, $1\le v\le 3$ **(2 pts)**, and the Jacobian is

$$\begin{vmatrix}\frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}}\\ \frac{1}{2\sqrt{uv}} & -\frac{1}{2}\frac{\sqrt{u}}{v\sqrt{v}}\end{vmatrix}=-\frac{1}{2v}.\ \textbf{(2 pts)}$$

Thus, The integral is changed into

$$Area=\int_1^2\int_1^3\frac{\sqrt{2}}{2v}\,dvdu\ \textbf{(2 pts)}=\frac{\ln 3}{\sqrt{2}}.\ \textbf{(2 pts)}$$

Method.2 Let $r(r,\theta) = (r\cos\theta, r\sin\theta, r)$, $\left(\frac{2}{\sin 2\theta}\right)^{\frac{1}{2}} \le r \le \left(\frac{4}{\sin 2\theta}\right)^{\frac{1}{2}}$, $\cot^{-1} 3 \le \theta \le \cot^{-1} 1$ **(3 pts)**. First compute

$$\begin{aligned}
&r_r(r,\theta) = (\cos\theta, \sin\theta, 1), \ \textbf{(1 pt)} \\
&r_\theta(r,\theta) = (-r\sin\theta, r\cos\theta, 0), \ \textbf{(1 pt)} \\
&|r_r(r,\theta) \times r_\theta(r,\theta)| = |(-r\cos\theta, -r\sin\theta, r)| = \sqrt{2}r. \ \textbf{(2 pts)}
\end{aligned}$$

Thus, the area is

$$\begin{aligned}
\int_{\cot^{-1}3}^{\cot^{-1}1} \int_{\left(\frac{2}{\sin 2\theta}\right)^{\frac{1}{2}}}^{\left(\frac{4}{\sin 2\theta}\right)^{\frac{1}{2}}} \sqrt{2}r\,drd\theta \ \textbf{(2 pts)} &= \sqrt{2}\int_{\cot^{-1}3}^{\cot^{-1}1} \frac{1}{2}\left(\frac{4}{\sin 2\theta} - \frac{2}{\sin 2\theta}\right) d\theta \ \textbf{(2 pts)} \\
&= \sqrt{2}\int_{\cot^{-1}3}^{\cot^{-1}1} \csc 2\theta\, d\theta \\
&= -\frac{1}{\sqrt{2}}\left[\ln(1+0) - \ln(\frac{5}{3} + \frac{4}{3})\right] \ \textbf{(2 pts)} \\
&= \frac{\ln 3}{\sqrt{2}}. \ \textbf{(2 pts)}
\end{aligned}$$

3. Let $\boldsymbol{F}(x,y) = (M(x,y), N(x,y))$ have continuous first and second derivatives everywhere in $\mathbb{R}^2$ and let $R = \{x^2 + y^2 < 1, x > 0, y > 0\}$.

   (a) (10 pts) State Green's Theorem in both forms for $\boldsymbol{F}$ on R.

   (b) (16 pts) Take $\boldsymbol{F} = (y, x)$ and verify both forms on R. That is, evaluate both line integral and double integral and check that they are the same. Do this for both forms.

Answer.

(a)

$$\begin{aligned}
\oint_C \boldsymbol{F}\cdot\mathbf{n}ds = \oint_C Mdy - Ndx = \iint_R (M_x + N_y)\,dxdy \ \textbf{(5 pts)} \\
\oint_C \boldsymbol{F}\cdot\mathbf{T}ds = \oint_C Mdx + Ndy = \iint_R (N_x - M_y)\,dxdy \ \textbf{(5 pts)}
\end{aligned}$$

where C is a piecewise smooth, simple closed curve which encloses the region R.

(b) Let $r_1(t) = (\cos t, \sin t), 0 \le t \le \frac{\pi}{2}$; $r_2(t) = (0, 1-t), 0 \le t \le 1$; $r_3(t) = (t, 0), 0 \le t \le 1$

(4 pts). Then

$$\begin{aligned}
\oint_C \boldsymbol{F}\cdot\mathbf{n}\,ds &= \int_0^{\frac{\pi}{2}}(\sin t\cos t-\cos t(-\sin t))\,dt+\int_0^1((1-t)\cdot(-1)-0\cdot 0)\,dt\\
&+\int_0^1(0\cdot 0-t\cdot 1)\,dt\\
&= 1-\frac{1}{2}-\frac{1}{2}=0=\iint_R(0+0)\,dxdy=\iint_R(M_x+N_y)\,dxdy,\ \textbf{(6 pts)}\\
\oint_C \boldsymbol{F}\cdot\mathbf{T}\,ds &= \int_0^{\frac{\pi}{2}}(\sin t(-\sin t)+\cos t\cos t)\,dt+\int_0^1((1-t)\cdot 0+0\cdot(-1))\,dt\\
&+\int_0^1(0\cdot 1+t\cdot 0)\,dt\\
&= 0+0+0=\iint_R(1-1)\,dxdy=\iint_R(N_x-M_y)\,dxdy.\ \textbf{(6 pts)}
\end{aligned}$$

4. (10 pts) True or false? Give details.

   If $f(x,y,z)$ has continuous first derivatives in a domain D, and $C=\{(x(t),y(t),z(t)),\ 0\le t\le 1\}$ be a smooth curve in D. Then $\int_C \nabla f\cdot \boldsymbol{T}ds$ depends only on f, $(x(0),y(0),z(0))$ and $(x(1),y(1),z(1))$.

   Answer. True. **(2 pts)**

$$\begin{aligned}
&\int_C \nabla f\cdot\mathbf{T}\,ds\\
=&\int_0^1(f_x(x(t),y(t),z(t))x'(t)+f_y(x(t),y(t),z(t))y'(t)+f_z(x(t),y(t),z(t))z'(t))\,dt\ \textbf{(4 pts)}\\
=&\int_0^1\partial_t(f(x(t),y(t),z(t))\,dt=f(x(1),y(1),z(1))-f(x(0),y(0),z(0)).\ \textbf{(4 pts)}
\end{aligned}$$

5. (24 pts) Let $R=\{1/4\le x^2+y^2\le 4\}$, $\boldsymbol{F}(x,y)=(2y,x)$, $\boldsymbol{G}(x,y)=(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}})$, $\boldsymbol{H}(x,y)=(\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2})$.

   Which one(s) of $\boldsymbol{F}$, $\boldsymbol{G}$ and $\boldsymbol{H}$ is (are) conservative on R? (That is, which one(s) of $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$, $\int_C \boldsymbol{G}\cdot d\boldsymbol{r}$ and $\int_C \boldsymbol{H}\cdot d\boldsymbol{r}$ is (are) zero on every closed loop C in R?) Explain.

   Answer.

   $\boldsymbol{F}$ is not conservative **(2 pts)** since $\partial_y(2y)=2\neq 1=\partial_x(x)$ **(2 pts)**.

   $\boldsymbol{H}$ is not conservative **(2 pts)** since for some $r(t)=(a\cos t,a\sin t)$, $0\le t\le 2\pi$ **(2 pts)**,

$$\oint_C \boldsymbol{H}\cdot\boldsymbol{T}\,ds=\int_0^{2\pi}\left(-\frac{\sin t}{a},\frac{\cos t}{a}\right)\cdot(-a\sin t,a\cos t)\,dt=\int_0^{2\pi}1\,dt=2\pi\neq 0.\ \textbf{(6 pts)}$$

   $\boldsymbol{G}$ is conservative. **(2 pts)** There are two methods to check it.

Method.1 It suffices to check whether $\oint_C \boldsymbol{G}\cdot\boldsymbol{T}\,ds = 0$ for any loop of the form $C = \{x^2+y^2 = a^2,\ 1/2 \le a \le 2\}$ since $\boldsymbol{G}$ satisfies the component test.

$$\partial_y\left(\frac{x}{\sqrt{x^2+y^2}}\right) = -\frac{xy}{(x^2+y^2)^{3/2}} = \partial_x\left(\frac{y}{\sqrt{x^2+y^2}}\right) \textbf{ (2 pts)}.$$

Let $r(t) = (a\cos t, a\sin t)$, $0 \le t \le 2\pi$.

$$\oint_C \boldsymbol{G}\cdot\boldsymbol{T}\,ds = \int_0^{2\pi} (\cos t, \sin t)\cdot(-a\sin t, a\cos t)\,dt = \int_0^{2\pi} 0\,dt = 0. \textbf{ (6 pts)}$$

Method.2 Find $f = f(x,y)$ such that $\boldsymbol{G} = \nabla f$.

$$\begin{aligned} & f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \\ \Rightarrow\quad & f(x,y) = \sqrt{x^2+y^2} + g(y) \\ \Rightarrow\quad & f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}} + g'(y) = \frac{y}{\sqrt{x^2+y^2}} \\ \Rightarrow\quad & g(y) = C \\ \Rightarrow\quad & f(x,y) = \sqrt{x^2+y^2} + C. \textbf{ (8 pts)} \end{aligned}$$

6. (15 pts) Let $\boldsymbol{F}(x,y,z) = z\boldsymbol{k}$ and $S = \{x^2+y^2+z^2 = 4,\ x>0,\ y>0,\ z>0\}$. Evaluate $\int\int_S \boldsymbol{F}\cdot\boldsymbol{n}\,d\sigma$ where $\boldsymbol{n}$ is unit normal of S pointing away from the origin.

Answer.

Method.1 Let $r(\phi,\theta) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$, $0 \le \phi, \theta \le \frac{\pi}{2}$ **(3 pts)**. Then

$$\begin{aligned} r_\phi(\phi,\theta) &= (2\cos\phi\cos\theta, 2\cos\phi\sin\theta, -2\sin\phi) \textbf{ (1 pt)} \\ r_\theta(\phi,\theta) &= (-2\sin\phi\sin\theta, 2\sin\phi\cos\theta, 0) \textbf{ (1 pt)} \\ r_\phi \times r_\theta &= (4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\cos\phi\sin\phi). \textbf{ (2 pts)} \end{aligned}$$

Therefore,

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,d\sigma = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} (0,0,2\cos\phi)\cdot(4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\cos\phi\sin\phi)\,d\phi d\theta \textbf{ (4 pts)}$$

$$= 8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos^2\phi\sin\phi\,d\phi d\theta = 4\pi\cdot\int_0^1 u^2\,du \textbf{ (2 pts)} = \frac{4\pi}{3}. \textbf{ (2 pts)}$$

Method.2 Let $g(x,y,z) = x^2+y^2+z^2 = 4$. Then

$$\nabla g = (2x, 2y, 2z),\ |\nabla g\cdot p| = 2z. \textbf{ (7 pts)}$$

$$\begin{aligned}
\iint_R \boldsymbol{F}\cdot\boldsymbol{n}\,d\sigma &= \iint_R \boldsymbol{F}\cdot\frac{\nabla g}{|\nabla g|}\frac{|\nabla g|}{|\nabla g\cdot p|}\,dxdy \\
&= \iint_R \boldsymbol{F}\cdot\frac{\nabla g}{|\nabla g\cdot p|}\,dxdy \ \textbf{(4 pts)} \\
&= \iint_R z\,dxdy \ \textbf{(2 pts)} \\
&= \text{the volume of the sphere of radius 2 in the first octant} \\
&= \frac{4\pi\cdot 2^3}{3}\cdot\frac{1}{8} \\
&= \frac{4\pi}{3}.\ \textbf{(2 pts)}
\end{aligned}$$

7. (10 pts) Let $f(x,y) = 2x + 3y + 4 + \left(\sqrt{x^2+y^2}\right)^{\frac{3}{2}}$. Is $f(x,y)$ differentiable at $(0,0)$? Explain.
   Answer. Yes. **(2 pts)**
   First we compute

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{2h+h^3}{h} = 2 \ \textbf{(2 pts)}$$
$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{3h+h^3}{h} = 3.\ \textbf{(2 pts)}$$

   Given $(\triangle x, \triangle y)$. Then

$$f(\triangle x,\triangle y) - f(0,0) = 2\triangle x + 3\triangle y + \left(\sqrt{\triangle x^2+\triangle y^2}\right)^{\frac{3}{2}}.\textbf{(2 pts)}$$

   To show $\left(\sqrt{\triangle x^2+\triangle y^2}\right)^{\frac{3}{2}} = \epsilon_1\triangle x + \epsilon_2\triangle y$. **(2 pts)**

Method.1 Write $(\triangle x,\triangle y) = (r\cos\theta, r\sin\theta)$. Then

$$\left(\sqrt{\triangle x^2+\triangle y^2}\right)^{\frac{3}{2}} = r^{\frac{3}{2}} = r\cdot r^{\frac{1}{2}} = r(\epsilon_1\cos\theta + \epsilon_2\sin\theta) = \epsilon_1\triangle x + \epsilon_2\triangle y$$

for some ϵ_1, ϵ_2 which approach to zero as $\triangle x, \triangle y \to 0$.

Method.2

$$\left(\sqrt{\triangle x^2+\triangle y^2}\right)^{\frac{3}{2}} = \left(\sqrt{\triangle x^2+\triangle y^2}\right)^{\frac{1}{2}}\sqrt{\triangle x^2+\triangle y^2} = o(1)\sqrt{\triangle x^2+\triangle y^2} \overset{HW7}{=} \epsilon_1\triangle x + \epsilon_2\triangle y$$

for some ϵ_1, ϵ_2 which approach to zero as $\triangle x, \triangle y \to 0$.

8. (10 pts) Evaluate $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n+1}$ on $|x|<1$ using computational rules of power series.
   Answer.

For $x = 0$, clearly $\sum_{n=0}^{\infty} \frac{x^n}{n+1} = 1$.
For $0 < |x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \textbf{ (4 pts)}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln(1-x) \textbf{ (4 pts)}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} \frac{x^n}{n+1} = -\frac{\ln(1-x)}{x}. \textbf{ (2 pts)}$$