Brief answers to Quiz 4

Apr 28, 2016

1. **(20 pts)** Find a tangent vector to the curve given by the intersection of the two surfaces $xyz = 1$ and $x^2 + 2y^2 + 3z^2 = 6$ at the point $(1, 1, 1)$.
Answer. Let $f(x, y, z) = xyz - 1$ and $g(x, y, z) = x^2 + 2y^2 + 3z^2$. Then

$$\nabla f(1, 1, 1) = (yz, xz, xy)|_{(1,1,1)} = (1, 1, 1), \ \textbf{(8 pts)}$$

$$\nabla g(1, 1, 1) = (2x, 4y, 6z)|_{(1,1,1)} = (2, 4, 6). \ \textbf{(8 pts)}$$

Thus

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = (1, -2, 1). \ \textbf{(4 pts)}$$

is a tangent vector.

2. **(20 pts)** Find all critical points of $f(x, y) = x^2 + xy + y^2 - 6x + 2$ and determine whether they are local min, local max or neither.
Answer.

$$\nabla f(x, y) = (2x + y - 6, x + 2y) = (0, 0) \ \textbf{(6 pts)} \Rightarrow (x, y) = (4, -2). \ \textbf{(4 pts)}$$

$$f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = 1 \Rightarrow f_{xx} > 0,\ f_{xx}f_{yy} - f_{xy}^2 > 0 \ \textbf{(6 pts)} \Rightarrow \text{ local minimum. } \textbf{(4 pts)}$$

3. **(20 pts)** Use the method of Lagrangian multiplier (only) to find the maximum and minimum value of $f(x, y, z) = x - 2y + 5z$ on the sphere $x^2 + y^2 + z^2 = 30$.
Answer. Let $g(x, y, z) = x^2 + y^2 + z^2 - 30$. Solve

$$\begin{aligned}
& \nabla f = \lambda \nabla g \\
\Rightarrow\quad & (1, -2, 5) = \lambda(2x, 2y, 2z) \ \textbf{(4 pts)} \\
\Rightarrow\quad & x = \frac{1}{2\lambda},\ y = -\frac{1}{\lambda},\ z = \frac{5}{2\lambda} \\
\Rightarrow\quad & \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{25}{4\lambda^2} = 30 \\
\Rightarrow\quad & \lambda = \pm\frac{1}{2} \ \textbf{(4 pts)} \\
\Rightarrow\quad & (x, y, z) = (1, -2, 5),\ (-1, 2, -5) \ \textbf{(4 pts)} \\
\Rightarrow\quad & f(1, -2, 5) = 30 \text{ is the maximum } \textbf{(4 pts)},\ f(-1, 2, -5) = -30 \text{ is the minimum } \textbf{(4 pts)}.
\end{aligned}$$

4. **(20 pts)** Use Taylor's formula to find the quadratic approximation of $f(x, y) = \dfrac{1}{1 - x - y}$ near the origin.
Answer.

$$f_x(x, y) = f_y(x, y) = \frac{1}{(1 - x - y)^2} \ \textbf{(4 pts)}$$

$$f_{xx}(x,y) = f_{xy}(x,y) = f_{yy}(x,y) = \frac{2}{(1-x-y)^3} \text{ (4 pts)}$$

$$\begin{aligned} Q(x,y) &= f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2) \text{ (8 pts)} \\ &= 1 + x + y + \frac{1}{2}(2x^2 + 4xy + 2y^2) \text{ (4 pts)} \end{aligned}$$

5. **(20 pts)** Let $U = f(P,V,T)$ where P, V and T are subject to the constraint $PV = nRT$, n, R are constants. Find $\left(\frac{\partial U}{\partial P}\right)_V$ and $\left(\frac{\partial U}{\partial T}\right)_V$

Answer.

$$\left(\frac{\partial U}{\partial P}\right)_V = \frac{\partial U}{\partial P} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial P} \text{ (6 pts)} = \frac{\partial U}{\partial P} + \frac{\partial U}{\partial T}\left(\frac{V}{nR}\right) \text{ (4 pts)}$$

$$\left(\frac{\partial U}{\partial T}\right)_V = \frac{\partial U}{\partial P}\frac{\partial P}{\partial T} + \frac{\partial U}{\partial T} \text{ (6 pts)} = \frac{\partial U}{\partial P}\left(\frac{nR}{V}\right) + \frac{\partial U}{\partial T} \text{ (4 pts)}$$