Brief answers to Quiz 3

Apr 14, 2016

1. **(20 pts)** Give formal definitions of (1): $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$, and (2): The function $z = f(x,y)$ is differentiable at (x_0, y_0).
   Answer.
   (1) Textbook p.774. **(10 pts)**
   (2) Textbook p.790. **(10 pts)**

2. **(20 pts)** Does $\lim_{(x,y)\to(0,0)} \dfrac{x^3 - xy^2}{x^2+y^2}$ exist? Explain.
   Answer.
   Yes. **(4 pts)**
   $0 \leq \left|\dfrac{r^3(\cos^3\theta - \cos\theta\sin^2\theta)}{r^2}\right|$ **(4 pts)** $= r|\cos^3\theta - \cos\theta\sin^2\theta|$ **(4 pts)** $\leq 2r$ **(4 pts)** $\to 0$ as $r \to 0$. Therefore, the limit exists by Sandiwich Principle. **(4 pts)**

3. **(20 pts)** Let $f(x,y) = \dfrac{xy^2}{x^2+y^4}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Do $f_x(0,0)$ and $f_y(0,0)$ exist? Explain. Is f continuous at $(0,0)$? Explain.
   Answer.
   Yes. **(2 pts)**
   $$\lim_{t\to0}\frac{f(t,0)-f(0,0)}{t} = \lim_{t\to0}\frac{0}{t} = 0 \Rightarrow f_x(0,0) = 0; \text{ } \mathbf{(4\ pts)}$$
   $$\lim_{t\to0}\frac{f(0,t)-f(0,0)}{t} = \lim_{t\to0}\frac{0}{t} = 0 \Rightarrow f_y(0,0) = 0. \text{ } \mathbf{(4\ pts)}$$
   No. **(2 pts)**
   $$\lim_{t\to0} f(t,0) = \lim_{t\to0}\frac{0}{t^2} = 0; \text{ } \mathbf{(4\ pts)}$$
   $$\lim_{t\to0} f(t^2,t) = \lim_{t\to0}\frac{t^4}{2t^4} = \frac{1}{2}. \text{ } \mathbf{(4\ pts)}$$

4. **(20 pts)** Let $F(x) = \int_0^x \cos(x^2 - t^2)dt$. Evaluate $F'(0)$. Give details.
   Answer.
   $$F(x) = G(x, y(x)) \text{ where } G(x,y) = \int_0^x \cos(y^2-t^2)\,dt \text{ and } y(x) = x \text{ } \mathbf{(8\ pts)}$$
   $$\Rightarrow \quad F'(x) = \cos(x^2-x^2) - \int_0^x \sin(x^2-t^2)\cdot 2x\,dt \text{ } \mathbf{(8\ pts)}$$
   $$\Rightarrow \quad F'(0) = 1. \text{ } \mathbf{(4\ pts)}$$

5. **(20 pts)** Let $f(x,y) = x^2 - xy + 2y^2$. Find the direction $\boldsymbol{u}$ (a unit vector) for which the directional derivative $\left(\dfrac{df}{ds}\right)_{\boldsymbol{u},(1,1)}$ (that is, $D_{\boldsymbol{u}}f(1,1)$), is largest.

Answer.

$\nabla f(1,1) = (2x-y, -x+4y)|_{(1,1)}$ **(12 pts)** $= (1,3)$ **(4 pts)** $\Rightarrow u = (1/\sqrt{10}, 3/\sqrt{10})$ **(4 pts)**.