# Quiz 2

Mar 17, 2016

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1. **(20 pts)** State (need not prove) the Leibnitz test for alternating series. Then give an example of $a_n$ such that $\sum_n a_n$ converges and $\sum_n |a_n|$ diverges.
   **Answer.**
   Suppose $a_n \geq 0$ **(3 pts)**, $a_n$ is decreasing **(3 pts)**, and $a_n \to 0$ **(3 pts)**. Then $\sum_n (-1)^n a_n$ **(3 pts)** converges **(3 pts)**.

   Let $a_n = (-1)^n/n$. **(5 pts)** Then $\sum_n a_n$ converges and $\sum_n |a_n|$ diverges.

2. **(20 pts)** Give an example of a power series that converges on $[0, 2]$ and diverges elsewhere. Explain. Do the same for $(3, 7]$.
   **Answer.**
   Consider
   $$\sum_n \frac{(-1)^n}{n}(x-1)^{2n}. \text{ } (\mathbf{4pts})$$
   Then
   $$\lim_{n\to\infty} \frac{n|x-1|^{2(n+1)}}{(n+1)|x-1|^{2n}} = |x-1|^2 < 1 \Leftrightarrow x \in (0,2). \text{ } (\mathbf{2pts})$$
   For $x = 0$,
   $$\sum_n \frac{(-1)^n}{n} \text{ converges by Leibnitz Test. } (\mathbf{2pts})$$
   For $x = 2$,
   $$\sum_n \frac{(-1)^n}{n} \text{ converges for the same reason. } (\mathbf{2pts})$$
   Thus, this series converges exactly on $[0, 2]$.

   Consider
   $$\sum_n \frac{(-1)^n}{n}\left(\frac{x-5}{2}\right)^n. \text{ } (\mathbf{4pts})$$
   Then
   $$\lim_{n\to\infty} \frac{n}{n+1}\left|\frac{x-5}{2}\right| = \left|\frac{x-5}{2}\right| < 1 \Leftrightarrow x \in (3,7). \text{ } (\mathbf{2pts})$$
   For $x = 3$,
   $$\sum_n \frac{1}{n} \text{ diverges by P-Test. } (\mathbf{2pts})$$

For $x = 7$,

$$\sum_n \frac{(-1)^n}{n} \text{ converges by Leibnitz Test. } (\mathbf{2pts})$$

Thus, this series converges exactly on $(3, 7]$.

3. **(20 pts)** Give the power series representation (centered at $a = 0$) of $\frac{1}{1+x}$ and find its radius of convergence. Then use it to find the power series representation (centered at $a = 0$) of $\ln(1 + x)$.
**Answer.**

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \ (\mathbf{5pts}),\ R = 1.\ (\mathbf{5pts})$$

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\, dt\ (\mathbf{5pts}) = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}.\ (\mathbf{5pts})$$

4. **(20 pts)** Suppose we know the power series representation of $\sin x$ (centered at $a = 0$), $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ is valid for all $x \in \mathbb{R}$. Use it to find the power series representation of $\cos x$ (centered at $a = 0$). Next, assume that $\tan x = \sum_{n=0}^{\infty} c_n x^n$ on $-R < x < R$, find $c_0, \cdots, c_5$.
**Answer.**

$$\cos x = (\sin x)' = \left( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right)' = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.\ (\mathbf{8pts})$$

$$P_5(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5.$$

$$\Rightarrow c_0 = 0,\ c_1 = 1,\ c_2 = 0, c_3 = \frac{1}{3},\ c_4 = 0,\ c_5 = \frac{2}{15}.\ (\mathbf{12pts})$$

5. **(20 pts)** True or False? Explain.

If $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ converges on $(a-1, a+1)$, then $a_n = \frac{f^{(n)}(a)}{n!}$.
**Answer.**
True. **(4 pts)**

$$f^{(n)}(x) = a_n n! + a_{n+1}(n+1)\ldots 2(x-a) + a_{n+2}(n+2)\ldots 3(x-a)^2 + \ldots\ (\mathbf{16pts})$$

$$\Rightarrow f^{(n)}(a) = a_n n!.$$