Solutions to selected problems in HW for Week 15

1. Section.16.3: Problem 26.
   Check the component test conditions:

$$\partial_y\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = -\frac{yz}{(x^2+y^2+z^2)^{3/2}} = \partial_z\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\partial_x\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) = -\frac{xy}{(x^2+y^2+z^2)^{3/2}} = \partial_y\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\partial_x\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = -\frac{xz}{(x^2+y^2+z^2)^{3/2}} = \partial_z\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)$$

   Since the natural domain $D = \{(x,y,z) \in \mathbb{R}^3 : \ (x,y,z) \neq (0,0,0)\}$ is simply connected, we can conclude that the line integral is independt of path.

2. Section.16.4: Problem 10.
   Let $F = Mi + Nj$ and R the region enclosed by the curve C. Since M and N have continuous first partial derivatives everywhere, we can apply Green's Theorem.

$$\oint_C F \cdot T\, ds = \iint_R (N_x - M_y)\, dxdy \iint_R (2-3)\, dxdy = (-1)Area(R) = -\pi\sqrt{2}$$
$$\oint_C F \cdot n\, ds = \iint_R (M_x + N_y)\, dxdy = \iint_R (1-1)\, dxdy = 0$$

3. Section.16.4: Problem 38.
   Let $F = Mi + Nj$ and C an arbitrary piecewise smooth simple closed curve enclosing a region R. Since M and N have continuous first partial derivatives everywhere, we can apply Green's Theorem.

$$\oint_C F \cdot T\, ds = \iint_R 1 - \left(\frac{1}{4}x^2 + y^2\right)\, dxdy$$

   which admits the maximal value as $R = \{(x,y) : \ 1 - \frac{1}{4}x^2 + y^2 \geq 0\}$, that is, as C is the curve $\frac{1}{4}x^2 + y^2 = 1$.

4. Section.16.4: Problem 39.

a. First compute $\nabla f = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right)$. Since the components of ∇f are not defined at $(0,0)$, we cannot apply Green's Theorem on the region enclosed by any circle $C : x^2 + y^2 = a^2$, $a > 0$. Thus we compute the line integral directly. Let $r(t) = (a\cos t, a\sin t)$, $0 \le t \le 2\pi$. (counterclockwise)

$$\oint_C \nabla f \cdot n\, ds = \int_0^{2\pi} \left(\frac{2\cos t}{a}, \frac{2\sin t}{a}\right) \cdot (a\cos t, a\sin t)\, dt = 4\pi.$$

b. Let R be the region enclosed by K.
If $(0,0)$ lies ouside K, then the components of ∇f have continuous derivaives at $(0,0)$, and thus we can apply Green's Theorem.

$$\oint_K \nabla f \cdot n\, ds = \iint_R \partial_x\left(\frac{2x}{x^2+y^2}\right) + \partial_y\left(\frac{2y}{x^2+y^2}\right) dxdy$$

$$= \iint_R \frac{2(y^2-x^2)}{(x^2+y^2)^2} + \frac{2(x^2-y^2)}{(x^2+y^2)^2}\, dxdy = 0$$

If $(0,0)$ lies inside K, then we choose a circle curve $C : x^2 + y^2 = a^2$ with $a > 0$ small enough so that C totally lies inside K (no intersection). Then the new region R enclosed by these two curves K and C doesn't contain $(0,0)$, and thus we can apply Green's Theorem.

$$\oint_K \nabla f \cdot n\, ds \;\circlearrowleft + \oint_C \nabla f \cdot n\, ds \;\circlearrowright$$

$$= \oint_K \left(\frac{2x}{x^2+y^2}dy - \frac{2y}{x^2+y^2}dx\right) \circlearrowleft + \oint_C \left(\frac{2x}{x^2+y^2}dy - \frac{2y}{x^2+y^2}dx\right) \circlearrowright$$

$$= \iint_R \partial_x\left(\frac{2x}{x^2+y^2}\right) + \partial_y\left(\frac{2y}{x^2+y^2}\right) = 0$$

$$\Rightarrow \oint_K \nabla f \cdot n\, ds + (-4\pi) = 0$$

$$\Rightarrow \oint_K \nabla f \cdot n\, ds = 4\pi$$

5. Homework assignment: Problem 2.
Let $D = \{(x,y,z) \in \mathbb{R}^3 : \; x^2 + y^2 \neq 0\}$.

(a) Check directly.

(b)

$$\begin{aligned}
\partial_z f = 0 \quad & \Rightarrow f(x,y,z) = g(x,y) \\
& \Rightarrow \partial_x f = \partial_x g = \frac{x}{\sqrt{x^2+y^2}} \\
& \Rightarrow g(x,y) = \sqrt{x^2+y^2} + h(y) \\
& \Rightarrow \partial_y f = \frac{y}{\sqrt{x^2+y^2}} + h'(y) = \frac{y}{\sqrt{x^2+y^2}} \\
& \Rightarrow h(y) = C \\
& \Rightarrow f(x,y,z) = \sqrt{x^2+y^2} + C
\end{aligned}$$

Thus F is conservative on D.

(c) Let $r(t) = (\cos t)i + (\sin t)j$, $0 \le t \le 2\pi$. Then

$$\int_0^{2\pi} G(r(t)) \cdot r'(t)\, ds = \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = 2\pi \neq 0.$$

Thus G is not conservative on D.

(d) We can determine whether H is conservative on D just by checking whether

$$(*) \quad \oint_{\{(x,y,z_0):\, x^2+y^2=d^2\}} H \cdot T\, ds = 0$$

for any $z_0 \in \mathbb{R}$ and $d \ge 0$. To see this, we need to verify that $\oint_K H \cdot T\, ds = 0$ for any loop K which surrounds the z-axis from our assumption $(*)$. For other loop K which doesn't surround the z-axis, we can apply Stokes' Theorem on the piecewise smooth oriented surface enclosed by K to conclude that

$$\oint_K H \cdot T\, ds = \iint_S \nabla \times H \cdot n = 0$$

where the last equality comes from the component test conditions.

Now let K be a loop which surrounds the z-axis. Let $C = \{(x,y,z_0) :\ x^2+y^2 = d^2\}$ be chosen well so that C and K form the boundary of a "two-sided" piecewise smooth oriented surface S. Then we can apply Stokes' Theorem on S:

$$\oint_K H \cdot T\, ds + \oint_C H \cdot T\, ds = \iint_S \nabla \times H \cdot n = 0$$

where the last equality comes from the component test conditions. By our assumption $(*)$, $\oint_C H \cdot T\, ds = 0$. Thus,

$$\oint_K H \cdot T\, ds = 0.$$

6. Homework assignment: Problem 3.

(1) $D = \{(x, y, z) \in \mathbb{R}^3 : \ (x, y, z) \neq (0, 0, 0)\} = \mathbb{R}^3 \setminus \{(0, 0, 0)\}$.

(2) Check directly.

(3) Yes.

(4) Yes. Since D is simply connected (by (3)) and F satisfies the component test conditions on D (by (2)), F is conservative on D. In fact, one after integration, it is easy to find the $F = \nabla f$ where $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.