Solutions to selected problems in HW for Week 09

1. Section 14.6: Problem 66.
   Check that $\gamma(1) = (1, 1, -1)$ satsfies $x^2 + y^2 - z = 3$. Then compute

$$\nabla f(1, 1, -1) = (2x, 2y, -1)|_{(1,1,-1)} = (2, 2, -1)$$

$$\gamma'(1) = (1/(2\sqrt{t}), 1/(2\sqrt{t}), -1/4)|_{t=1} = (1/2, 1/2, -1/4)$$

   which are parallel.

2. Section 14.7: Problem 44.
   (a) $f(x, y) = x^2y^2 \geq 0 = f(0, 0)\ \forall\ (x, y) \in \mathbb{R}^2 \Rightarrow f$ has a local minimum at $(0, 0)$.
   (b) $f(x, y) = 1 - x^2y^2 \leq 1 = f(0, 0)\ \forall\ (x, y) \in \mathbb{R}^2 \Rightarrow f$ has a local maximum at $(0, 0)$.
   (c) Note that f is differentiable and $\nabla f(0, 0) = (y^2, 2xy)|_{(0,0)} = (0, 0)$. $f(1, 1) = 1 > 0 = f(0, 0)$ and $f(-1, 1) = -1 < 0 = f(0, 0) \Rightarrow f$ has a saddle point at $(0, 0)$.
   (d) Note that f is differentiable and $\nabla f(0, 0) = (3x^2y^2, 2x^3y)|_{(0,0)} = (0, 0)$. $f(1, 1) = 1 > 0 = f(0, 0)$ and $f(-1, 1) = -1 < 0 = f(0, 0) \Rightarrow f$ has a saddle point at $(0, 0)$.
   (e) Note that f is differentiable and $\nabla f(0, 0) = (3x^2y^3, 3x^3y^2)|_{(0,0)} = (0, 0)$. $f(1, 1) = 1 > 0 = f(0, 0)$ and $f(-1, 1) = -1 < 0 = f(0, 0) \Rightarrow f$ has a saddle point at $(0, 0)$.
   (f) $f(x, y) = x^4y^4 \geq 0 = f(0, 0)\ \forall\ (x, y) \in \mathbb{R}^2 \Rightarrow f$ has a local minimum at $(0, 0)$.

3. Homework assignment: Problem 4.
   Write $X_0 = (x_0, y_0, z_0)$, $X = (x, y, z)$. Check that

$$\begin{aligned}
&F(t) = f(X_0 + t\triangle X),\ 0 \leq t \leq 1 \\
&F'(t) = f_x(X_0 + t\triangle X)\triangle x + f_y(X_0 + t\triangle X)\triangle y + f_z(X_0 + t\triangle X)\triangle z \\
&\vdots \\
&F^{(n)}(t) = [(\triangle x\partial_x + \triangle y\partial_y + \triangle z\partial_z)^n f](X_0 + t\triangle X) \\
\Rightarrow\quad &F(t) = F(0) + \sum_{k=1}^{n} \frac{1}{k!}F^{(k)}(0)t^k + \frac{1}{(n+1)!}F^{(n+1)}(c)t^{n+1} \text{ for some } c \text{ between 0 and } t \\
\Rightarrow\quad &f(X_0 + \triangle X) = F(1) = F(0) + \sum_{k=1}^{n} \frac{1}{k!}F^{(k)}(0) + \frac{1}{(n+1)!}F^{(n+1)}(c) \\
&\text{(for some } 0 < c < 1\text{)}
\end{aligned}$$

$$= f(X_0) + \sum_{k=1}^{n} \frac{1}{k!}[(\triangle x\partial_x + \triangle y\partial_y + \triangle z\partial_z)^k f](X_0) + \frac{1}{(n+1)!}[(\triangle x\partial_x + \triangle y\partial_y + \triangle z\partial_z)^{n+1} f](X_0 + c\triangle X).$$