Solutions to selected problems in HW for Week 07

1. Section 14.3: Problem 60.

$$\partial_x f(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\sin h^3}{h^3} = 1.$$
$$\partial_y f(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\sin h^4}{h^3} = \lim_{h\to 0} h\cdot\frac{\sin h^4}{h^4} = 0.$$

2. Section 14.3: Problem 67.

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$2a = 2bc\sin A\partial_a A \Rightarrow \partial_a A = \frac{a}{bc\sin A}$$
$$0 = 2b - 2c\cos A + 2bc\sin A\partial_b A \Rightarrow \partial_b A = \frac{c\cos A - b}{bc\sin A}$$

3. Section 14.4: Problem 51.
Let
$$G(u,x) = \int_0^u \sqrt{t^4+x^3}\,dt \text{ and } u = x^2.$$
Then
$$F'(x) = G_u(x^2,x)\cdot 2x + \int_0^u \frac{3x^2}{2\sqrt{t^4+x^3}}\,dt = \sqrt{x^8+x^3}\cdot 2x + \int_0^{x^2} \frac{3x^2}{2\sqrt{t^4+x^3}}\,dt.$$

4. Show that if $f(x,y) = o(1)|x-x_0| + o(1)|y-y_0|$ as $(x,y)\to(x_0,y_0)$ then $f(x,y) = o(1)\cdot\sqrt{(x-x_0)^2+(y-y_0)^2}$ and vice versa (the converse). Note that all three $o(1)$ refer to 2D limits as $(x,y)\to(x_0,y_0)$.
Proof. Let $\triangle x = |x-x_0|$ and $\triangle y = |y-y_0|$.

" $\Rightarrow$ "
$$\frac{f(x,y)}{\sqrt{\triangle x^2+\triangle y^2}} = o(1)\cdot\frac{\triangle x}{\sqrt{\triangle x^2+\triangle y^2}} + o(1)\cdot\frac{\triangle y}{\sqrt{\triangle x^2+\triangle y^2}} = o(1)$$
since
$$0 \le o(1)\cdot\frac{\triangle x}{\sqrt{\triangle x^2+\triangle y^2}} \le o(1) \to 0$$
$$0 \le o(1)\cdot\frac{\triangle y}{\sqrt{\triangle x^2+\triangle y^2}} \le o(1) \to 0$$
as $(x,y)\to(x_0,y_0)$.

" $\Leftarrow$ "

$$f(x,y) = o(1)\sqrt{\triangle x^2 + \triangle y^2} = o(1) \cdot \frac{\triangle x}{\sqrt{\triangle x^2 + \triangle y^2}} \triangle x + o(1) \cdot \frac{\triangle y}{\sqrt{\triangle x^2 + \triangle y^2}} \triangle y$$

$$= o(1)\triangle x + o(1)\triangle y$$

as $(x,y) \to (x_0, y_0)$ by the same argument as above.

5. Suppose that $F(x,y,z) = 0$ can implicitly define $x = f(y,z)$, or $y = g(z,x)$, or $z = h(x,y)$ near some point (x_0, y_0, z_0) with $F(x_0, y_0, z_0) = 0$. (for example, $F(x,y,z) = x + 2y + 3z4$ can). Show that, for any such point (x_0, y_0, z_0), we have

$$f_y g_z h_x = f_z g_x h_y = -1.$$

Proof.

$$F(f(y,z), y, z) = 0 \Rightarrow F_x f_y + F_y = 0 \Rightarrow f_y = -\frac{F_y}{F_x}$$
$$F(x, g(z,x), z) = 0 \Rightarrow F_y g_z + F_z = 0 \Rightarrow g_z = -\frac{F_z}{F_y}$$
$$F(x, y, h(x,y)) = 0 \Rightarrow F_x + F_z h_x = 0 \Rightarrow h_x = -\frac{F_x}{F_z}$$

$$\Rightarrow f_y g_z h_x = -1.$$

Simialr to another equality.