Solutions to selected problems in HW for Week 05

1. Section 10.9: Problem 41.
   By The Remainder Estimation Theorem, $\exists\, c$ between 0 and x such that
   $$|error| \le e^c \frac{|x|^3}{3!} \le e^{0.1}\frac{(0.1)^3}{6}.$$

2. Section 10.9: Problem 42.
   By Alternating Series Estimation Theorem,
   $$|error| \le \frac{|x|^3}{3!} \le \frac{(0.1)^3}{6}.$$

3. Section 10.9: Problem 50a.
   Let $p = \pi + x$, then $|x| < 10^{-n}$.
   $$p + \sin p = \pi + x + \sin(\pi + x) = \pi + x - \sin x = \pi + \frac{x^3}{3!} + O(x^5)$$
   $$\Rightarrow |p + \sin p - \pi| \le \frac{|x|^3}{6} + O(|x|^5) < \frac{1}{6}10^{-3n} + O(10^{-5n}) < \frac{1}{6}10^{-3n} + \frac{1}{2}10^{-3n} < 10^{-3n}.$$

4. Section 10.10: Problem 17.
   By Alternating Series Estimation Theorem,
   $$|error| \le \int_0^{0.1} \left|\binom{-\frac{1}{2}}{n} x^{4n}\right| dx = \left|\binom{-\frac{1}{2}}{n}\right| \left|\int_0^{0.1} x^{4n}\, dx = \left|\binom{-\frac{1}{2}}{n}\right| \right| \frac{(0.1)^{4n+1}}{4n+1} < 10^{-3}$$
   It follows that $n = 1$ is enough. Therefore, the approximation is
   $$\int_0^{0.1} 1\, dx = 0.1.$$

5. Section 10.10: Problem 19.
   By Alternating Series Estimation Theorem,
   $$|error| \le \int_0^{0.1} \frac{x^{2n}}{(2n+1)!}\, dx = \frac{(0.1)^{2n+1}}{(2n+1)!(2n+1)} < 10^{-8}$$
   . It follows that $n = 3$ is enough. Therefore, the approximation is
   $$\int_0^{0.1} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right) dx = 0.099944461111111.$$

6. Section 10.10: Problem 27.
   Apply Alternating Series Estimation Theorem.
   (a)

$$|error| \leq \frac{x^{2n+2}}{(2n+1)(2n+2)} = \frac{(0.5)^{2n+2}}{(2n+1)(2n+2)} < 10^{-3} \Rightarrow n = 2.$$

   Therefore, the approximation is

$$\frac{x^2}{2} - \frac{x^4}{12}.$$

   (a)

$$|error| \leq \frac{x^{2n+2}}{(2n+1)(2n+2)} = \frac{1}{(2n+1)(2n+2)} < 10^{-3} \Rightarrow n = 16.$$

   Therefore, the approximation is

$$\sum_{n=0}^{15} \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)}.$$

7. Section 10.10: Problem 64.
   (a)

$$\begin{aligned}(1+x)f'(x) &= (1+x)\sum_{k\geq 1}\binom{m}{k}kx^{k-1} = \binom{m}{1} + \sum_{k\geq 1}\left[\binom{m}{k+1}(k+1) + \binom{m}{k}k\right]x^k \\ &= \binom{m}{1} + \sum_{k\geq 1}\left[k\binom{m+1}{k+1} + \binom{m}{k+1}\right]x^k \\ &= m + m\sum_{k\geq 1}\binom{m}{k}x^k = mf(x).\end{aligned}$$

   (b) $g'(x) = -m(1 = x)^{-m-1}f(x) + (1+x)^{-m}f'(x) = 0$.
   (c) g is constant. $g(x) = g(0) = f(0) = 1 \Rightarrow f(x) = (1+x)^m$.

8. Section 10.10: Problem 66.
   For $|t| > 1$,

$$\frac{1}{1+t^2} = \frac{1}{t^2}\frac{1}{1+\frac{1}{t^2}} = \frac{1}{t^2}\left(1 - \frac{1}{t^2} + \frac{1}{t^4} + \dots\right) = \frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{t^6} + \dots.$$

   For $x > 1$,

$$\frac{\pi}{2} - \tan^{-1} x = \int_x^\infty \frac{1}{1+t^2}\,dt = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \dots.$$

   For $x < -1$,

$$\tan^{-1} x + \frac{\pi}{2} = \int_{-\infty}^x \frac{1}{1+t^2}\,dt = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \dots.$$

9. Section 10.10: Problem 74.

$$\begin{aligned}
& \int e^{ax}(\cos bx + i\sin bx)\,dx \\
= \ & \int e^{(a+ib)x}\,dx \\
= \ & \frac{a-ib}{a^2+b^2}e^{(a+ib)x} + C_1 + iC_2 \\
= \ & \frac{e^{ax}}{a^2+b^2}\,(a\cos bx + b\sin bx) + C_1 \\
+ \ & i\left[\frac{e^{ax}}{a^2+b^2}\,(a\sin bx - b\cos bx) + C_2\right].
\end{aligned}$$