Solutions to selected problems in HW for Week 04

1. Let
$$f(x) = \begin{cases} 0, & x = 0; \\ e^{-1/x^2}, & x \neq 0. \end{cases}$$
It is known that $f^{(n)}(0) = 0$ for all n. Verify this for $f'(0)$ and $f''(0)$.
Proof.
$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-1/x^2}}{x} = \lim_{t\to\infty} \frac{e^{-t}}{1/\sqrt{t}} = 0 \text{ (L'Hôspital's Rule)}$$
$$\Rightarrow f'(0) = 0.$$
Check that
$$f'(x) = \begin{cases} 0, & x = 0; \\ 2x^{-3}e^{-1/x^2}, & x \neq 0. \end{cases}$$
$$\lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} \frac{2x^{-3}e^{-1/x^2}}{x} = 2\lim_{t\to\infty} \frac{e^{-t}}{1/t^2} = 0 \text{ (L'Hôspital's Rule)}$$
$$\Rightarrow f''(0) = 0.$$