Solutions to selected problems in HW for Week 01

1. Section 8.7: Problem 35.
   $\int_0^{\pi/2} \tan\theta\, d\theta = \lim_{c\to\pi/2-} \int_0^c \tan\theta\, d\theta = \lim_{c\to\pi/2-} \int_{\cos c}^1 \frac{du}{u} = \lim_{c\to\pi/2-} \ln u|_{\cos c}^1 = \infty.$

2. Section 8.7: Problem 65.

   (a) $\int_1^2 \frac{dx}{x(\ln x)^p} = \int_0^{\ln 2} \frac{du}{u^p}$ converges for $p < 1$ and diverges for $p \geq 1$.

   (b) $\int_2^\infty \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^\infty \frac{du}{u^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$.

3. Section 8.7: Evaluate
   $$\int_0^1 \frac{1}{x^p}\, dx = \lim_{\epsilon\to 0} \int_\epsilon^1 \frac{1}{x^p}\, dx$$
   for $0 < p < 1$, $p = 1$, and $p > 1$, respectively.

$p \neq 1$: $\int_\epsilon^1 \frac{1}{x^p}\, dx = \frac{1}{1-p} x^{1-p}|_\epsilon^1 = \frac{1}{1-p}(1-\epsilon^{1-p}) \to \begin{cases} \infty & \text{if } p > 1 \\ \frac{1}{1-p} & \text{if } 0 < p < 1 \end{cases}$ as $\epsilon \to 0+$.

$p = 1$: $\int_\epsilon^1 \frac{1}{x}\, dx = \ln x|_\epsilon^1 = -\ln\epsilon \to \infty$ as $\epsilon \to 0+$.

4. Section 10.3: Problem 53. (Cauchy Condensation Test) Suppose that a_n is positive and $a_n \searrow 0$. Then
   $$\sum_n a_n \text{ con.} \Leftrightarrow \sum_n 2^n a_{2^n} \text{ con.}$$

   Proof. Let $s_n = a_1 + \ldots + a_n$ and $t_m = a_1 + 2a_2 + \ldots + 2^m a_{2^m}$.

"$\Rightarrow$"

$$\begin{aligned} a_1 + \ldots + a_{2^m} &= a_1 + a_2 + (a_3 + a_4) + \ldots + (a_{2^{m-1}+1} + \ldots + a_{2^m}) \\ &\geq a_1 + a_2 + (a_4 + a_4) + \ldots + (a_{2^m} + \ldots + a_{2^m}) \\ &= \frac{1}{2}a_1 + \frac{1}{2}t_m \\ \Rightarrow t_m &\leq 2\left(s_{2^m} - \frac{1}{2}a_1\right) \leq 2\left(\sum_n a_n - \frac{1}{2}a_1\right) =: M' \end{aligned}$$

That implies $\{t_m\}$ is an increasing and bounded from above, and thus converges.

" $\Leftarrow$ "

$$
\begin{aligned}
a_1 + \ldots + a_{2^m} &= a_1 + (a_2 + a_3) + \ldots + (a_{2^{m-1}} + \ldots + a_{2^m - 1}) + a_{2^m} \\
&\leq a_1 + (a_2 + a_2) + \ldots + (a_{2^{m-1}} + \ldots + a_{2^{m-1}}) + a_{2^m} \\
&= t_{m-1} + a_{2^m} \\
\Rightarrow s_{2^m - 1} &\leq t_{m-1} \\
\Rightarrow s_n &\leq s_{2^m - 1} \leq t_{m-1} \leq \sum_m 2^m a_{2^m} =: M \quad \text{for some } m.
\end{aligned}
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That implies $\{s_n\}$ is an increasing and bounded from above, and thus converges.