Midterm Exam 2

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1. (10 pts) True or False? If true, prove it. If false, give a counter example.

   If $|f(x) - (3x+2)| \le |x|^{1.5}$ for all $x \in R$, then f is differentiable at $x = 0$.

   Ans:
   True. **(2 points)**
   Take $x = 0$, then $|f(0) - 2| \le 0 \Rightarrow f(0) = 2$. **(2 points)**
   Consider $|\frac{f(x)-f(0)}{x} - 3| = |\frac{f(x)-(3x+2)}{x}| \le |x|^{0.5} \to 0$ as $x \to 0$. **(6 points)**

2. (a) (6 pts) Graph $f(x) = \dfrac{x}{\sqrt{x^2+1}}$. Give all details including possible asymptotes.

   (b) (6 pts) The function $y = f(x)$ is odd ($f(-x) = -f(x)$) and the root x^* to the equation $f(x) = 0$ is $x^* = 0$. Give formula of Newton's method for finding this root.

   (c) (6 pts) The Newton's method does not always converge. There is an $a > 0$ such that Newton's method converges if and only if $-a < x_0 < a$. Take this fact for granted and find a (show how to find a, but need NOT prove that Newton's method converge if and only if $-a < x_0 < a$).

   Ans:

   (a) $f'(x) = \frac{1}{(x^2+1)^{3/2}}$. **(1 point)** $f''(x) = \frac{-3x}{(x^2+1)^{5/2}}$. **(1 point)**
   Graphing-
   Inflection point: $(0, 0)$ **(1 point)**
   Increasing on: $(-\infty, \infty)$ **(1 point)**
   Concave up on: $(-\infty, 0)$; concave down on: $(0, \infty)$ **(1 point)**
   $f(x) \to \pm 1$ as $x \to \pm\infty \Rightarrow y = \pm 1$ are horizontal asymptotes. **(1 point)**

   (b) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ **(2 points)** $= x_n - \frac{\frac{x_n}{\sqrt{x_n^2+1}}}{\frac{1}{(x_n^2+1)^{3/2}}}$ **(2 points)** $= -x_n^3$ **(2 points)**.

   (c) $|x_n - x_*| = |x_n| = |x_{n-1}|^3 = \ldots = |x_0|^{3^n} \to 0$ **(3 points)** if and only if $|x_0| < 1$. Thus $a = 1$ **(3 points)**.

3. (12 pts) Let f be a differentiable function defined on $\{x \ge 0\}$ satisfying

   (a): $f(0) = -1$,

   (b): $f'(x) \ge 1/2$ for all $x \ge 0$.

   Show that $f(x) = 0$ has <u>one and only one</u> solution on $\{x \ge 0\}$.

   Ans:

   Step 1:

$f(x) = 0$ has <u>one</u> solution on $\{x \geq 0\}$:

Since

$$f(2) = f(0) + \int_0^2 f'(x)dx \geq -1 + (2-0) \min_{0\leq x\leq 2} f'(x) \geq -1 + 2 \times \frac{1}{2} = 0 \ \ \textbf{(2points)}$$

and $f(0) = -1 < 0$. **(1 point)**

Moreover, $f'(x)$ exist for all $x \geq 0$, which implies $f(x)$ is continuous on $x \geq 0$ **(1 point)**. By Intermediate Value Theorem, there exists c between 0 and 2 such that $f(c) = 0$ **(2 points)**.

Step 2:

$f(x) = 0$ has <u>only one</u> solution on $\{x \geq 0\}$:

If there exist $c_1 \geq 0$, $c_1 \neq c$ such that $f(c_1) = 0$ **(2 points)**, by Rolle's Theorem, there exist c_2 between c and c_1 such that

$$f'(c_2) = 0 \ \ \textbf{(2points)}$$

which is contradict to $f'(x) \geq \frac{1}{2}$ for all $x \geq 0$. Hence $f(x) = 0$ has only one solution on $x \geq 0$ **(2 points)**.

4. (18 pts) Find the limits of the following expressions:

$$\text{(a)} \quad \lim_{x\to 0^+} x^x \qquad \text{(b)} \quad \lim_{x\to 0^+} \frac{e^{\frac{-1}{x}}}{x} \qquad \text{(c)} \quad \lim_{x\to 0} \frac{x^2 \cos\frac{1}{x}}{\sin x}$$

Ans:

(a) $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x}$ **(2points)** $= e^{\lim_{x\to 0^+} x\ln x}$ **(2points)** $= 1$ **(2points)**.

(b) $\lim_{x\to 0^+} \frac{e^{\frac{-1}{x}}}{x} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{e^{\frac{1}{x}}}$ **(2points)** $= \lim_{x\to 0^+} \frac{1}{e^{\frac{1}{x}}}$ **(2points)** $= 0$ **(2points)**.

(c) $\lim_{x\to 0} \frac{x^2\cos\frac{1}{x}}{\sin x} = \lim_{x\to 0} \frac{x}{\sin x} \lim_{x\to 0} x\cos\frac{1}{x}$ **(2points)** $= 1\cdot 0$ **(2points)** $= 0$ **(2points)**.

5. (8 pts) Solve for $y(x)$ on $x < 0$ from

$$y''(x) = x^{-2}, \quad y(-1) = 1, \quad y'(-1) = 2.$$

Ans:

$\int_{-1}^{x} y''(t)\,dt = \int_{-1}^{x} t^{-2}\,dt = -\frac{1}{x} - 1$ **(2 points)** $\Rightarrow y'(x) = 1 - \frac{1}{x}$ **(2 points)**.

$\int_{-1}^{x} y'(t)\,dt = \int_{-1}^{x} (1 - \frac{1}{t})\,dt = -\ln|x| + x + 1$ **(2 points)** $\Rightarrow y(x) = -\ln|x| + x + 2$ **(2 points)**.

6. (8 pts) Evaluate $\lim_{n\to\infty} \sum_{k=n}^{2n} \frac{n}{k^2}$

Ans:

$$\lim_{n\to\infty} \sum_{k=n}^{2n} \frac{n}{k^2} = \lim_{n\to\infty} (\frac{n}{n^2} + \sum_{k=n+1}^{2n} \frac{1}{n} \frac{1}{(\frac{k}{n})^2})\textbf{(3points)} = \int_1^2 \frac{1}{x^2}dx\textbf{(3points)} = \frac{1}{2}\textbf{(2points)}$$

7. (14 pts) State both parts of Fundamental Theorem of Calculus, prove that 'part 1 implies part 2'. If you can't prove this, you could prove 'part 1' instead.

Ans:

The Fundamental Theorem of Calculus, Part 1:

If f is continuous on $[a, b]$**(2 points)**, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$**(2 points)**:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x) \tag{1}$$

The Fundamental Theorem of Calculus, Part 2:

If f is continuous at every point in $[a, b]$**(2 points)** and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(t)dt = F(b) - F(a)(\mathbf{2points}) \tag{2}$$

Proof of part 1 implies part 2:

Let $G(x)$ be any antiderivative of $f(x)$, which is

$$\frac{d}{dx}G(x) = f(x)$$

Since $\frac{d}{dx}\int_a^x f(t)dt = f(x) = \frac{d}{dx}G(x)$, we have

$$\int_a^x f(t)dt + c = G(x)$$

for some constant $c \in R$.**(3 points)**

Then $G(b) - G(a) = \int_a^b f(t)dt + c - (\int_a^a f(t)dt + c) = \int_a^b f(t)dt$**(3 points)**.

8. (12 pts) Evaluate

$$\text{(a)} \quad \int_1^2 \frac{1}{x(1+\ln^2 x)}dx \qquad \text{(b)} \quad \int_0^4 x\sqrt{2x+1}\,dx$$

Ans: (a) Let $u = \ln x$

$$\int_1^2 \frac{1}{x(1+\ln^2 x)}dx = \int_0^{\ln 2} \frac{1}{1+u^2}du(\mathbf{3points}) = \tan^{-1}(\ln 2)(\mathbf{3points})$$

(b) Let $z = 2x + 1$

$$\int_0^4 x\sqrt{2x+1}\,dx = \int_1^9 \frac{1}{4}(z-1)\sqrt{z}dz(\mathbf{3points}) = \frac{121}{5} - \frac{13}{3} = \frac{298}{15}(\mathbf{3points})$$

9. True or False? If true, prove it. If false, give a counter example.

   (a) (3 pts) If $y = f(x)$ is differentiable at $x = c$ then it is continuous at $x = c$.

   (b) (3 pts) If $y = f(x)$ is continuous at $x = c$ then it is differentiable at $x = c$.

   Ans:

   (a) True. Suppose $f'(c)$ exitst.

   $$\lim_{x\to c}[f(x)-f(c)] = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot(x-c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot\lim_{x\to c}(x-c) = f'(c)\cdot 0 = 0.$$

   (4 pts)

   (b) False. Let $f(x) = |x|$ and $c = 0$. Then f is continuous at c but not differentiable at c. $(\because \lim_{x\to 0+}\frac{|x|}{x} = 1 \neq -1 = \lim_{x\to 0-}\frac{|x|}{x})$

   (4 pts)

10. (8 pts) Start with domain and range for csc and $\csc^{-1}$, derive the formula for the derivative of $\csc^{-1}$.

   Ans:

   $$\csc y : [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}] \to (-\infty, -1] \cup [1, \infty),$$

   $$\csc^{-1} x : (-\infty, -1] \cup [1, \infty) \to [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]. \quad \textbf{(2 pts)}$$

   $$y = \csc^{-1} x \Rightarrow \csc y = x \Rightarrow -\csc y \cot y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = -\frac{1}{\csc y \cot y} \quad \textbf{(3 pts)}$$

   $$\Rightarrow \frac{dy}{dx} = \begin{cases} -\frac{1}{x\sqrt{x^2-1}}, & x > 1 \\ \frac{1}{x\sqrt{x^2-1}}, & x < -1 \end{cases} = -\frac{1}{|x|\sqrt{x^2-1}}, \ |x| > 1. \quad \textbf{(3 pts)}$$