Midterm Exam 1

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1. (10 pts) Evaluate $\lim_{\theta\to\frac{\pi}{3}} \frac{\cos\theta - \frac{1}{2}}{\theta - \frac{\pi}{3}}$.

   Ans:
$$\lim_{\theta\to\frac{\pi}{3}} \frac{\cos\theta - \frac{1}{2}}{\theta - \frac{\pi}{3}} = (\cos\theta)'|_{\theta=\frac{\pi}{3}} \textbf{(5 pts)} = -\sin\theta|_{\theta=\frac{\pi}{3}} = -\frac{\sqrt{3}}{2}. \textbf{(5 pts)}$$

2. (10 pts) Let $f_i(x) = a_i x^2 + b_i x + c_i$, $i = 1, 2, 3$. Evaluate
$$\frac{d}{dx}\begin{vmatrix} f_1(x) & f_1'(x) & f_1''(x) \\ f_2(x) & f_2'(x) & f_2''(x) \\ f_3(x) & f_3'(x) & f_3''(x) \end{vmatrix}$$

   Ans:
$$= \begin{vmatrix} f_1'(x) & f_1'(x) & f_1''(x) \\ f_2'(x) & f_2'(x) & f_2''(x) \\ f_3'(x) & f_3'(x) & f_3''(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_1''(x) & f_1''(x) \\ f_2(x) & f_2''(x) & f_2''(x) \\ f_3(x) & f_3''(x) & f_3''(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_1'(x) & f_1'''(x) \\ f_2(x) & f_2'(x) & f_2'''(x) \\ f_3(x) & f_3'(x) & f_3'''(x) \end{vmatrix} \quad \textbf{(6 pts)}$$
$$= \begin{vmatrix} f_1(x) & f_1'(x) & f_1'''(x) \\ f_2(x) & f_2'(x) & f_2'''(x) \\ f_3(x) & f_3'(x) & f_3'''(x) \end{vmatrix} \quad \textbf{(2 pts)}$$
$$= \begin{vmatrix} f_1(x) & f_1'(x) & 0 \\ f_2(x) & f_2'(x) & 0 \\ f_3(x) & f_3'(x) & 0 \end{vmatrix} = 0. \quad \textbf{(2 pts)}$$

3. (10 pts) Find $\frac{dy}{dx}$ where $y = x^{(x^x)}$, $x > 0$.

   Ans: $= x^{(x^x)} x^x (\ln x(1 + \ln x) + \frac{1}{x})$.**(3 pts+3 pts+4 pts)**

4. (10 pts) Find $y'(1)$ and $y''(1)$ where $y(x)$ is implicitly given by $\tan(x+y) + \sin^{-1}(x^2+y) = 0$ near $(x, y) = (1, -1)$.

   Ans:

   $\frac{d}{dx}$ once:
$$[\sec^2(x+y)](1+y') + \frac{2x+y'}{\sqrt{1-(x^2+y)^2}} = 0, \textbf{(3 pts)}$$
   evaluate at $x = 1, y = -1$, one gets $y'(1, -1) = -\frac{3}{2}$.**(2 pts)**

   $\frac{d}{dx}$ twice:

   $[2\sec^2(x+y)\tan(x+y)](1+y')^2 + [\sec^2(x+y)]y'' + \frac{(2+y'')(1-(x^2+y)^2)+(x^2+y)(2x+y')^2}{(1-(x^2+y)^2)^{\frac{3}{2}}} = 0$**(3 pts)**

   evaluate at $x = 1, y = -1$, one gets $y''(1, -1) = -1$.**(2 pts)**

5. (10 pts) Let f^{-1} be the inverse function of f. Evaluate $\dfrac{d^2}{dy^2}f^{-1}(y)$ in terms of f' and f''. Show all details.

Ans:

$$\frac{d}{dy}f^{-1}(y) = \frac{1}{f'(f^{-1}(y))}. \quad \textbf{(2 pts)}$$

$$\begin{aligned}\frac{d^2}{dy^2}f^{-1}(y) &= -\frac{f''(f^{-1}(y)) \cdot \frac{d}{dy}f^{-1}(y)}{[f'(f^{-1}(y))]^2} \quad \textbf{(6 pts)} \\ &= -\frac{f''(f^{-1}(y))}{[f'(f^{-1}(y))]^3}. \quad \textbf{(2 pts)}\end{aligned}$$

6. (10 pts) Let $f(x) : [0,1] \mapsto [0,1]$ be a continuous function. Prove that $f(x) = x$ has at least one solution.

Ans: Let $g(x) = f(x) - x$, then $g(x)$ is continuous on $[0,1]$.**(3 pts)**

Since $0 \leq f(x) \leq 1$, we have

$$\begin{aligned}g(0) &= f(0) - 0 \geq 0 \textbf{(2 pts)} \\ g(1) &= f(1) - 1 \leq 0 \textbf{(2 pts)}\end{aligned}$$

By Intermediate Value Theorem, $\exists c \in [0,1]$ s.t. $g(c) = 0$, i.e. $f(c) = c$.**(3 pts)**

Hence $f(x) = x$ has at least one solution on $[0,1]$.

7. (10 pts) Start with domain and range for csc and $\csc^{-1}$, derive the formula for the derivative of $\csc^{-1}$.

Ans:

$$\csc y : [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}] \to (-\infty, -1] \cup [1, \infty),$$

$$\csc^{-1} x : (-\infty, -1] \cup [1, \infty) \to [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]. \quad \textbf{(2 pts)}$$

$$y = \csc^{-1} x \Rightarrow \csc y = x \Rightarrow -\csc y \cot y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = -\frac{1}{\csc y \cot y} \quad \textbf{(4 pts)}$$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} -\frac{1}{x\sqrt{x^2-1}}, & x > 1 \\ \frac{1}{x\sqrt{x^2-1}}, & x < -1 \end{cases} = -\frac{1}{|x|\sqrt{x^2-1}}, \ |x| > 1. \quad \textbf{(4 pts)}$$

8. (10 pts) True or False? If true, prove it. If false, give a counter example.

   (a) If $y = f(x)$ is differentiable at $x = c$ then it is continuous at $x = c$.

   (b) If $y = f(x)$ is continuous at $x = c$ then it is differentiable at $x = c$.

Ans:

(a) True. Suppose $f'(c)$ exitst.

$$\lim_{x\to c}[f(x)-f(c)] = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot(x-c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot\lim_{x\to c}(x-c) = f'(c)\cdot 0 = 0.$$

(5 pts)

(b) False. Let $f(x) = |x|$ and $c = 0$. Then f is continuous at c but not differentiable at c. ($\because \lim_{x\to 0+}\frac{|x|}{x} = 1 \neq -1 = \lim_{x\to 0-}\frac{|x|}{x}$)
(5 pts)

9. (10 pts) Give formal definition of $\lim_{x\to\infty} f(x) = L$. Then verify that $\lim_{x\to\infty}\frac{1}{x} = 0$ using the $\varepsilon - \delta$ argument.

Ans:
$\lim_{x\to\infty} f(x) = L$ if given any $\epsilon > 0$, $\exists N > 0$ such that for all x, $x > N \Rightarrow |f(x)-L| < \epsilon$.**(5 pts)**

Given any $\epsilon > 0$, pick $N = \frac{1}{\epsilon} > 0$**(3 pts)**, then for all x, $x > N$, $|\frac{1}{x} - 0| = \frac{1}{x} < \frac{1}{N} = \epsilon$**(2 pts)**

10. (16 pts) Give formal definition of $\lim_{x\to c} f(x) \neq L$. Then verify that $\lim_{x\to 0}\sin x \neq 1$ using $\varepsilon - \delta$ argument.

Ans:
$\lim_{x\to c} f(x) \neq L$ if $\exists\ \epsilon > 0$ such that for any $\delta > 0$, there exists x such that $0 < |x-c| < \delta$ and $|f(x) - L| \geq \epsilon$. **(4 pts)**
Take $\epsilon = \frac{1}{2}$. Given any $\delta > 0$. Take $x = \min\{\frac{\delta}{2}, \frac{\pi}{6}\}$. **(6 pts)**
Then $0 < |x - 0| = x < \delta$ and $|\sin x - 1| = 1 - \sin x \geq 1 - \frac{1}{2} = \epsilon$. ($\because 0 < x \leq \frac{\pi}{6} \Rightarrow 0 < \sin x \leq \frac{1}{2}$) **(6 pts)**