Final Exam

Jan 12, 2016, 10:10AM

1. (12 pts) Find the solutions for

   (a) $\dfrac{dy}{dx} = 2x\sqrt{1-y^2},\ -1 < y < 1$

   (b) $\dfrac{dy}{dx} - y = -y^2$

   Ans:

   (1) $\int \frac{dy}{\sqrt{1-y^2}} = \int 2x\,dx \Rightarrow \sin^{-1} y = x^2 + C$ **(6 pts)** for some constant C.

   (2) $\frac{du}{dx} + u = 1$ **(2 pts)** $\Rightarrow (e^x u)' = e^x \Rightarrow e^x u = e^x + C \Rightarrow u = 1 + Ce^{-x} \Rightarrow y = \frac{1}{1+Ce^{-x}}$ **(4 pts)** for some constant C.

2. (12 pts) Find the volume and surface area of the object obtained by rotating the region $\{(x-3)^2 + y^2 \le 1,\ x \ge 3\}$ around the y axis. Note the surface area consists of two parts, one generated by a half circle, the other generated by a line segment.

   Ans:

   Volume $= \int_{-1}^{1} \pi((3+\sqrt{1-y^2})^2 - 3^2)\,dy$ (**3pts**) $= 6\pi \int_{-1}^{1} \sqrt{1-y^2}\,dy + \pi \int_{-1}^{1} (1-y^2)\,dy = 3\pi^2 + \frac{4}{3}\pi$.

   $\int_{-1}^{1} \sqrt{1-y^2}\,dy = \int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{\cos 2\theta + 1}{2}\,d\theta = \frac{\pi}{2}$. **(2 pts)**

   $\int_{-1}^{1} (1-y^2)\,dy = 2 - \frac{2}{3} = \frac{4}{3}$. **(1 pt)**

   Surface $= \int_{-1}^{1} 2\pi(3+\sqrt{1-y^2})\sqrt{\frac{1}{1-y^2}}dy$ (**3pts**) $+ 2\pi\cdot 3\cdot 2$ (**1pt**) $= 6\pi \int_{-1}^{1} \frac{1}{\sqrt{1-y^2}}\,dy + 2\pi \int_{-1}^{1} dy + 12\pi = 6\pi^2 + 16\pi$.

   $\int_{-1}^{1} \frac{1}{\sqrt{1-y^2}}\,dy = \sin^{-1} y\Big|_{-1}^{1} = \pi$. **(2 pts)**

3. (10 8pts) Order e^x, x^x, $(\ln x)^x$ and x^e from slowest to fastest growing rate as $x \to \infty$. Explain.

   Ans:

   $\lim_{x\to\infty} \dfrac{x^e}{e^x} = \lim_{x\to\infty} \dfrac{ex^{e-1}}{e^x} = \lim_{x\to\infty} \dfrac{e(e-1)x^{e-2}}{e^x} = \lim_{x\to\infty} \dfrac{e(e-1)(e-2)x^{e-3}}{e^x} = 0.$ (since $e < 3$) **(3 pts)**

   $\lim_{x\to\infty} \dfrac{e^x}{(\ln x)^x} = \lim_{x\to\infty} \dfrac{e^x}{e^{x\ln\ln x}} = \lim_{x\to\infty} \dfrac{1}{e^{x(\ln\ln x - 1)}} = 0.$ (since $\ln\ln x > 1$ for x is large enough) **(3 pts)**

   $\lim_{x\to\infty} \dfrac{(\ln x)^x}{x^x} = \lim_{x\to\infty} \dfrac{e^{x\ln\ln x}}{e^{x\ln x}} = \lim_{x\to\infty} \dfrac{1}{e^{x(\ln x - \ln\ln x)}} = 0.$ (since $\ln x > \ln\ln x$ for x is large enough) **(3 pts)**

   From slowest to fastest is x^e, e^x, $(\ln x)^x$, x^x. **(1 pts)**

4. (6 pts) Write down the form of partial fraction expansion for $\frac{x^7}{(1-x^4)^2}$. Need NOT find the undetermined coefficients.

Ans:
$\frac{x^7}{(1-x^4)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x} + \frac{D}{(1+x)^2} + \frac{Ex+F}{1+x^2} + \frac{Gx+H}{(1+x^2)^2}$.
where $A \sim H$ are undetermined coefficients.**(1 pts) for each term**

5. (50 pts)

$$(1) \quad \int_1^2 \frac{x}{\sqrt{2-x}} dx \quad (2) \quad \int_1^{e^\pi} \sin(\ln x)\, dx \quad (3) \quad \int_0^{\pi/4} \tan^3 x \sec^3 x\, dx$$

$$(4) \quad \int_1^2 \frac{1}{e^{2x} - e^{-x}}\, dx \quad (5) \quad \int \frac{1}{\sqrt{4x - x^2}}\, dx$$

Ans:

(1) Let $u^2 = 2 - x$

$$\int_1^2 \frac{x}{\sqrt{2-x}}\, dx = \int_1^0 \frac{2-u^2}{u}(-2u)\, du\textbf{(5 pts)} = \int_0^1 4 - 2u^2\, dx = (4u - \frac{2}{3}u^3)|_0^1 = \frac{10}{3}.\textbf{(5 pts)}$$

(2)

$$\int_1^{e^\pi} \sin(\ln x)\, dx = (x \sin(\ln x))|_1^{e^\pi} - \int_1^{e^\pi} \cos(\ln x)\, dx\textbf{(4 pts)} = -\int_1^{e^\pi} \cos(\ln x)\, dx$$

$$= \quad -[(x\cos(\ln x))|_1^{e^\pi} + \int_1^{e^\pi} \sin(\ln x)\, dx]\textbf{(4 pts)} = e^\pi + 1 - \int_1^{e^\pi} \sin(\ln x)\, dx$$

$$\Rightarrow \quad \int_1^{e^\pi} \sin(\ln x)\, dx = \frac{e^\pi + 1}{2}\textbf{(2 pts)}$$

(3) Let $u = \sec x$

$$\int_0^{\pi/4} \tan^3 x \sec^3 x\, dx = \int_1^{\sqrt{2}} (u^2 - 1)u^2\, du\textbf{(5 pts)} = (\frac{u^5}{5} - \frac{u^3}{3})|_1^{\sqrt{2}} = \frac{2 + 2\sqrt{2}}{15}\textbf{(5 pts)}$$

(4) Let $u = e^x$

$$\int_1^2 \frac{1}{e^{2x} - e^{-x}}\,dx = \int_1^2 \frac{e^x}{e^{3x}-1}\,dx = \int_e^{e^2} \frac{1}{u^3-1}\,du\textbf{(2 pts)} = \int_e^{e^2} \frac{1}{(u-1)(u^2+u+1)}\,du$$

$$= \int_e^{e^2} \frac{\frac{1}{3}}{u-1} + \frac{-\frac{1}{3}u - \frac{2}{3}}{u^2+u+1}\,du\textbf{(2 pts)}$$

$$= \int_e^{e^2} \frac{\frac{1}{3}}{u-1}\,du + \int_e^{e^2} \frac{-\frac{1}{3}u - \frac{1}{6}}{u^2+u+1}\,du + \int_e^{e^2} \frac{-\frac{1}{2}}{u^2+u+1}\,du$$

$$= [\frac{1}{3}\ln|u-1| - \frac{1}{6}\ln|u^2+u+1| - \frac{1}{\sqrt{3}}\tan^{-1}(\frac{2}{\sqrt{3}}(u+\frac{1}{2}))]|_e^{e^2}\textbf{(2 pts for each term)}$$

$$= \frac{1}{3}\ln(\frac{e^2-1}{e-1}) - \frac{1}{6}\ln(\frac{e^4+e^2+1}{e^2+e+1}) - \frac{1}{\sqrt{3}}(\tan^{-1}(\frac{2}{\sqrt{3}}(e^2+\frac{1}{2})) - \tan^{-1}(\frac{2}{\sqrt{3}}(e+\frac{1}{2})))$$

(5) Let $x - 2 = 2\sin\theta$**(5 pts)**

$$\int \frac{1}{\sqrt{4x-x^2}}\,dx = \int \frac{1}{\sqrt{4-(x-2)^2}}\,dx = \int \frac{1}{2\cos\theta} 2\cos\theta\,d\theta$$

$$= \int 1\,d\theta = \theta + C = \sin^{-1}(\frac{x-2}{2}) + C\textbf{(5 pts)}$$

for some constant C.

6. (10 pts) Express $\int_0^\pi \cos^6 x\ \sin^4 x\,dx$ in terms of $\int_0^\pi \cos^4 x\ \sin^4 x\,dx$.

Ans:

$$\int_0^\pi \cos^6 x\ \sin^4 x\,dx = \int_0^\pi \cos^5 x\ \sin^4 x \cos x\,dx$$

$$= [\cos^5 x(\frac{1}{5}\sin^5 x)]_0^\pi - \int_0^\pi (5\cos^4 x(-\sin x))(\frac{1}{5}\sin^5 x)\,dx\textbf{(4 pts)}$$

$$= \int_0^\pi \cos^4 x \sin^6 x\,dx = \int_0^\pi \cos^4 x \sin^4 x(1-\cos^2 x)\,dx$$

$$= \int_0^\pi \cos^4 x \sin^4 x\,dx - \int_0^\pi \cos^6 x \sin^4 x\,dx\textbf{(4 pts)}$$

$$\Rightarrow \int_0^\pi \cos^6 x\ \sin^4 x\,dx = \frac{1}{2}\int_0^\pi \cos^4 x \sin^4 x\,dx\textbf{(2 pts)}$$

7. (10 pts) Start with domain and range for sech and sech^{-1}, derive the formula for the derivative of sech^{-1}.

Ans:

$$\text{sech} : \mathbb{R} \to (0,1]\ \textbf{(2 pts)}$$

$$\text{sech}^{-1} : (0,1] \to [0,\infty)\ \textbf{(2 pts)}$$

$$
\begin{aligned}
\frac{d(\operatorname{sech}^{-1}x)}{dx} &= -\frac{1}{\operatorname{sech}(\operatorname{sech}^{-1}x)\tanh(\operatorname{sech}^{-1}x)} \textbf{ (2 pts)} \\
&= -\frac{1}{\operatorname{sech}(\operatorname{sech}^{-1}x)\sqrt{1-\operatorname{sech}^2(\operatorname{sech}^{-1}x)}} \textbf{ (2 pts)} \\
&\quad (\because\ x \in (0,1) \Rightarrow \operatorname{sech}^{-1}x > 0 \Rightarrow \tanh(\operatorname{sech}^{-1}x) > 0) \\
&= -\frac{1}{x\sqrt{1-x^2}} \textbf{ (2 pts)}.
\end{aligned}
$$

8. (8 pts) For what values of $p \in \mathbb{R}$, is the function $f(x) = |x|^p$ differentiable at $x = 0$? Explain.

   Ans:

   For $p > 1$,

   $$0 \le \left|\frac{|x|^p}{x}\right| = |x|^{p-1} \to 0$$

   as $x \to 0$. Thus, $|x|^p$ is differentiable at $x = 0$. **(3 pts)**

   For $p = 1$,

   $$\lim_{x\to 0+} \frac{|x|}{x} = 1 \neq -1 = \lim_{x\to 0-} \frac{|x|}{x}.$$

   Thus, $|x|^p$ is not differentiable at $x = 0$. **(2 pts)**

   For $p < 1$,

   $$\lim_{x\to 0+} \frac{|x|^p}{x} = \lim_{x\to 0+} \frac{1}{x^{1-p}} = +\infty.$$

   Thus, $|x|^p$ is not differentiable at $x = 0$. **(3 pts)**

9. (12 pts) State both parts of Fundamental Theorem of Calculus, prove that part 1 implies part 2.

   Ans:

   <u>Fundamental Theorem of Calculus Part 1</u>: **(3 pts)**

   > Suppose f is continuous on $[a, b]$.
   > Then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) with derivative
   >
   > $$F'(x) = f(x).$$

   <u>Fundamental Theorem of Calculus Part 1</u>: **(3 pts)**

   > Suppose f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$.
   > Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Part 1 implies Part 2: **(6 pts)**

Let $G(x) = \int_a^x f(t)\,dt$. Then by Part 1, G is an antiderivative of f. Given any antiderivative F of f, then

$$F(x) = G(x) + C, \quad x \in (a, b).$$

Since F and G are both continuous on $[a, b]$,

$$F(x) = G(x) + C, \quad x \in [a, b].$$

Therefore,

$$F(b) - F(a) = (G(b) + C) - (G(a) + C) = \int_a^b f(x)\,dx.$$