Quiz 5

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1. A torus (donut) is generated by revolving the disk $(x-2)^2+y^2 \leq 1$ around the y axis. Find the volume of the torus.

   Ans:
   Method I:(Method of disks)

$$\int_{-1}^{1} \pi[(2+\sqrt{1-y^2})^2-(2-\sqrt{1-y^2})^2]dy(\mathbf{10pts}) = \int_{-1}^{1} 8\pi\sqrt{1-y^2}dy = 8\pi \times \frac{\pi}{2} = 4\pi^2(\mathbf{10pts})$$

   Method II:(Method of cylinders)

$$\begin{aligned}
&\int_{1}^{3} 2\pi x[(\sqrt{1-(x-2)^2}-(-\sqrt{1-(x-2)^2})]dx(\mathbf{10pts}) = 4\pi\int_{1}^{3} x\sqrt{1-(x-2)^2}dx \\
= \;& 4\pi\int_{-1}^{1}(u+2)\sqrt{1-u^2}du \quad (\text{Let } u=x-2 \Rightarrow du=dx)(\mathbf{5pts}) \\
= \;& 4\pi\int_{-1}^{1} u\sqrt{1-u^2}du + 8\pi\int_{-1}^{1}\sqrt{1-u^2}du \\
= \;& 4\pi(\frac{-1}{3}(1-u^2)^{\frac{3}{2}})|_{-1}^{1} + 8\pi \times \frac{\pi}{2} = 0+4\pi^2 = 4\pi^2(\mathbf{5pts})
\end{aligned}$$

   Hence volume of the torus is $4\pi^2$.

2. Find the surface area of the torus in problem 1.

   Ans:

$$\int_{-1}^{1} 2\pi(2+\sqrt{1-y^2})\sqrt{1+(\frac{-y}{\sqrt{1-y^2}})^2}dy + \int_{-1}^{1} 2\pi(2-\sqrt{1-y^2})\sqrt{1+(\frac{y}{\sqrt{1-y^2}})^2}dy(\mathbf{10pts})$$

$$= 8\pi\int_{-1}^{1}\sqrt{1+(\frac{y^2}{1-y^2})}dy = 8\pi\int_{-1}^{1}\frac{1}{\sqrt{1-y^2}}dy = 8\pi\sin^{-1}y|_{-1}^{1} = 8\pi^2(\mathbf{10pts})$$

   Hence the surface area of the torus in problem is $8\pi^2$

3. Find the length of the curve $y = \int_0^x \sqrt{\cos t}\, dt$ from $x=0$ to $x=\frac{\pi}{2}$.

Ans:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sqrt{1+(\frac{dy}{dx})^2}dx &= \int_0^{\frac{\pi}{2}} \sqrt{1+\cos x}dx(\mathbf{10pts}) = \int_0^{\frac{\pi}{2}} \sqrt{2\cos^2\frac{x}{2}}dx(\mathbf{5pts}) \\ &= \sqrt{2}\int_0^{\frac{\pi}{2}} |\cos\frac{x}{2}|dx = \sqrt{2}\int_0^{\frac{\pi}{2}} \cos\frac{x}{2}dx = 2\sqrt{2}\sin\frac{x}{2}|_0^{\frac{\pi}{2}} = 2(\mathbf{5pts})\end{aligned}$$

Hence the length of the curve from $x = 0$ to $x = \frac{\pi}{2}$ is 2.

4. Find the solutions for $\frac{dy}{dx} = 3x^2e^{-y}$.

   Ans:

$$\begin{aligned}&\Rightarrow e^y\tfrac{dy}{dx} = 3x^2 \\ &\Rightarrow e^y = x^3 + C \quad (\mathbf{8pts}) \\ &\Rightarrow y = \ln(x^3 + C). \quad (\mathbf{12pts})\end{aligned}$$

5. Find the solution for $x\frac{dy}{dx} + y = e^x$ on $x > 0$ with $y(1) = 1$.

   Ans:

$$\begin{aligned}&\Rightarrow \tfrac{d(xy)}{dx} = e^x \\ &\Rightarrow xy = e^x + C \quad (\mathbf{8pts}) \\ &\Rightarrow y = \tfrac{e^x+C}{x} \\ &\Rightarrow y = \tfrac{e^x-e+1}{x} \text{ since } y(1) = 1. \quad (\mathbf{12pts})\end{aligned}$$