Quiz 4

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1. Find $\lim_{x\to 0}(\frac{1}{x} - \frac{1}{\sin x})$.

   Ans:

$$\begin{aligned} &\lim_{x\to 0}(\tfrac{1}{x} - \tfrac{1}{\sin x}) = \lim_{x\to 0} \tfrac{\sin x - x}{x \sin x} \textbf{(8 pts)} \\ = &\lim_{x\to 0} \tfrac{\cos x - 1}{\sin x + x\cos x} \textbf{(8 pts)} = \lim_{x\to 0} \tfrac{-\sin x}{2\cos x - x\sin x} \textbf{(4 pts)} = 0. \end{aligned}$$

2. Find the point on $y = \sqrt{x}$, $x \geq 0$ that is closest to $(2, 0)$. Explain why the answer you have is actually a global minimum.

   Ans:
   Let $D^2 = (x-2)^2 + (y-0)^2 = (x-2)^2 + x$ on $y = \sqrt{x}$ and $x \geq 0$.**(5 pts)**
   Then $\frac{dD^2}{dx} = 2x - 3$, hence $x = \frac{3}{2}$ is a critical point.**(5 pts)**
   Moreover $\frac{dD^2}{dx} < 0$ if $x < \frac{3}{2}$ and $\frac{dD^2}{dx} > 0$ if $x > \frac{3}{2}$.**(5 pts)**
   Hence D^2 has absolute minimum at $x = \frac{3}{2}$, and $(\frac{3}{2}, \sqrt{\frac{3}{2}})$ is the point on $y = \sqrt{x}$ which is closest to (2,0).**(5 pts)**

3. Write down Newton's method that can be used to find $\sqrt[3]{2}$. Need not give the numerical value.

   Ans:
   Let $f(x) = x^3 - 2$, then the root of $f(x) = 0$ gives $x = \sqrt[3]{2}$. Newtons' method: Start with a reasonable x_0, (for example $x_0 = 1$), then iterate $x_{n+1} = x_n - \frac{x_n^3 - 2}{3x_n^2}$.**(10 pts)**

4. Express $\displaystyle\int_1^2 \frac{1}{1+x^2}\, dx$ as a limit of Riemann sum (with uniformly spaced partition and c_k of your choice). Then find the limit of the definite integral using fundamental Theorem of Calculus.

   Ans:

   (1) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+c_k^2}\frac{1}{n}$ **(8 pts)** for some $c_k \in [\frac{k-1}{n}, \frac{k}{n}]$ **(2 pts)**.

   (2) Since $(\tan^{-1} x)' = \frac{1}{1+x^2}$ **(8 pts)**, by FTC we can know that the definite integral is $\tan^{-1}(2) - \tan^{-1}(1)$ **(2 pts)**.

5. Evaluate $\displaystyle\frac{d}{dx}\int_{x^2}^{0} \sqrt{1+t^4}\, dt$.

   Ans: $= -\frac{d}{dx}\int_0^{x^2} \sqrt{1+t^4}\, dt$ **(4 pts)** $= -\sqrt{1+x^8} \cdot \frac{d}{dx}(x^2)$ **(12 pts)** $= -\sqrt{1+x^8} \cdot (2x)$ **(4 pts)**.