Quiz 3

Nov 12, 2015

Show all details.

1. Find an approximate value of $\sqrt[3]{1.009}$ and show that the error of the approximation is smaller than 10^{-5}.

   Ans:
   (i)
   Let $f(x) = (1+x)^{\frac{1}{3}}$ and $L(x,0) = f(0) + f'(0)(x-0) = 1 + \frac{1}{3}x$**(10 points)**
   then $\sqrt[3]{1.009} = f(0.009) \approx L(0.009) = 1.003$**(5 points)**
   (ii)

$$|f(0.009) - L(0.009)| \leq \frac{1}{2} \max_{0 \leq c \leq 0.009} |f''(c)|(0.009-0)^2 = \frac{81}{2} \max_{0 \leq c \leq 0.009} |-\frac{2}{9}(1+x)^{-\frac{5}{3}}| \cdot 10^{-6}$$
$$= 9 \cdot 10^{-6} \leq 10^{-5}(\textbf{10points})$$

2. Show that, if Rolle's Theorem is true, then the Mean Value Theorem is true.

   Ans:
   If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b),
   let $g(x) = f(x) - [f(a) + \frac{f(b)-f(a)}{b-a}(x-a)]$.**(10 points)**
   Then $g(x)$ is also continuous on $[a,b]$ and differentiable on (a,b) with $g(a) = g(b)$.**(8 points)**
   Hence by Rolle's Theorem, $\exists c \in (a,b)$ such that $g'(c) = 0$,
   which implies $f'(c) = \frac{f(b)-f(a)}{b-a}$.**(7 points)**

3. Let $f(x) = x^{2/3}(3-x)$. Find absolute max and min of f on $[-3,3]$. Show all details.

   Ans:
   $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(3-x) - x^{\frac{2}{3}} = \frac{6-5x}{3x^{\frac{1}{3}}}$. **(6 points)**
   Critical points: $x = 0, \frac{6}{5}$. **(4 points)**
   $f(0) = f(3) = 0 < f(\frac{6}{5}) = \frac{9}{5} \cdot (\frac{6}{5})^{\frac{2}{3}} < f(-3) = 6 \cdot 3^{\frac{2}{3}}$. **(9 points)**
   Absolute maximum: $f(-3) = 6 \cdot 3^{\frac{2}{3}}$. **(3 points)**
   Absolute minimum: $f(0) = f(3) = 0$. **(3 points)**

4. Graph $y = \frac{1}{x^2-1}$. Show all details.

   Ans:
   Domain: $x \neq \pm 1$.
   $y' = \frac{-2x}{(x^2-1)^2}$, $x \neq \pm 1$. **(3 points)**
   $y'' = \frac{-2(x^2-1)^2 - (-2x)\cdot 2(x^2-1)2x}{(x^2-1)^4} = \frac{6x^2+2}{(x^2-1)^3}$, $x \neq \pm 1$. **(3 points)**

Increasing interval: $(-\infty, -1)$, $(-1, 0]$, Decreasing interval: $[0, 1)$, $(1, \infty)$. **(6 points)**
Concave up: $(-\infty, -1)$, $(1, \infty)$, Concave down: $(-1, 1)$. **(6 points)**
Critical points: $x = 0$.
Local maximum: $f(0) = -1$. **(1 point)**
Vertical asymptote: $\lim_{x \to \pm 1} y = \pm\infty \Rightarrow x = \pm 1$ are vertical asymptotes.
Horizontal asymptote: $\lim_{x \to \pm\infty} y = 0 \Rightarrow y = 0$ is a horizontal asymptote.
Graph: **(6 points)**