Brief answer to selected problems in HW00

1. Section 3.8: Problem 98. Let

$$f(x) = \ln x \Longrightarrow f'(x) = \frac{1}{x}$$

By definition, for fixed $x > 0$

$$\begin{aligned} f'(1) &= \lim_{\frac{x}{n}\to 0} \frac{f(1+\frac{x}{n})-f(1)}{\frac{x}{n}} = \lim_{n\to\infty} \frac{n}{x}(\ln(1+\frac{x}{n}) - \ln 1) \\ &= \frac{1}{x}\lim_{n\to\infty} \ln(1+\frac{x}{n})^n \end{aligned}$$

which implies

$$x = \ln(\lim_{n\to\infty}(1+\frac{x}{n})^n) \Longrightarrow \lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$$

2. Section 3.9: Problem 55. After differentiation, we find out

$$f'(x) = g'(x), \quad x > 0$$

which means

$$f(x) = g(x) + c, \quad x \geq 0$$

where c is some constant. And $x = 0$ gives us

$$-\frac{\pi}{2} = 0 + c, \quad x \geq 0$$

hence

$$f(x) = g(x) - \frac{\pi}{2}, \quad x \geq 0$$

On the other hand, we also can derive same result by following process. For $x \geq 0$, let

$$\theta = \sin^{-1}\frac{x-1}{x+1} \Longrightarrow \sin\theta = \frac{x-1}{x+1}$$

then

$$\tan(\frac{\theta+\frac{\pi}{2}}{2}) = \sqrt{\frac{1-\cos(\theta+\frac{\pi}{2})}{1+\cos(\theta+\frac{\pi}{2})}} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sqrt{x}$$

hence

$$\tan^{-1}\sqrt{x} = \frac{\theta+\frac{\pi}{2}}{2} \Longrightarrow \sin^{-1}\frac{x-1}{x+1} = 2\tan^{-1}\sqrt{x} - \frac{\pi}{2}$$

3. Hw05: Problem 3. Domain and range for csc and $\csc^{-1}$

$$\begin{aligned}
\csc \quad & : \{\theta|\theta \in \mathbf{R}, \theta \neq k\pi, \forall k \in \mathbf{Z}\} \mapsto (-\infty, -1] \cup [1, \infty) \\
\text{restricted } \csc \quad & : [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}] \mapsto (-\infty, -1] \cup [1, \infty) \\
\csc^{-1} \quad & : (-\infty, -1] \cup [1, \infty) \mapsto [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]
\end{aligned}$$

For $|x| > 1$, let

$$\begin{aligned}
\theta = \csc^{-1} x \quad & \Longrightarrow \quad \csc\theta = x \\
& \Longrightarrow \quad -\csc\theta \cot\theta \frac{d\theta}{dx} = 1 \\
& \Longrightarrow \quad \frac{d\theta}{dx} = \frac{-1}{\csc\theta\cot\theta}
\end{aligned}$$

since

$$\cot\theta \begin{cases} > 0 & , \theta \in (0, \frac{\pi}{2}) \text{ or } x > 1 \\ < 0 & , \theta \in (-\frac{\pi}{2}, 0) \text{ or } x < -1 \end{cases}$$

we have

$$\cot\theta = \begin{cases} \sqrt{x^2 - 1} & , x > 1 \\ -\sqrt{x^2 - 1} & , x < -1 \end{cases}$$

hence

$$\frac{d \csc^{-1} x}{dx} = \frac{-1}{|x|\sqrt{x^2 - 1}}, |x| > 1$$