Brief solutions to Midterm Exam 2

1. (10 pts) Solve for $y(x)$ on $x < 0$ from

$$y''(x) = x^{-2}, \quad y(-1) = 1, \quad y'(-1) = 2$$

Ans:

$$y(x) = -\ln|x| + x + 2$$

2. (12 pts) Graph $f(x) = x^{1/3}(x-4)$. Indicate all critical points and points of inflection.

Ans:

$f'(x) = \frac{4}{3}x^{-\frac{2}{3}}(x-1)$ and $f''(x) = \frac{4}{9}x^{-\frac{5}{3}}(x+2)$. We have

$$\begin{cases} \text{Critical points: x=0, 1} \\ \text{Points of inflection: x=0, -2} \end{cases}$$

3. (16 pts) Find the limits of the following expressions:

$$\text{(a)} \quad \lim_{x\to 0^+} x^x \qquad \text{(b)} \quad \lim_{x\to 0} \frac{x^2 \cos\frac{1}{x}}{\sin x}$$

Ans:

$$\text{(a)} \lim_{x\to 0^+} x^x = 1 \quad \text{(b)} \lim_{x\to 0} \frac{x^2\cos\frac{1}{x}}{\sin x} = \lim_{x\to 0} \frac{x}{\sin x} x \cos\frac{1}{x} = 0$$

4. (16 pts) Evaluate

$$\text{(a)} \quad \int_1^2 \frac{1}{x(1+\ln^2 x)} dx \qquad \text{(b)} \quad \int_0^4 x\sqrt{2x+1}\, dx$$

Ans: (a) Let $u = \ln x$

$$\int_1^2 \frac{1}{x(1+\ln^2 x)} dx = \int_0^{\ln 2} \frac{1}{1+u^2} du = \tan^{-1}(\ln 2)$$

(b) Let $z = 2x + 1$

$$\int_0^4 x\sqrt{2x+1}\, dx = \int_1^9 \frac{1}{4}(z-1)\sqrt{z} dz = \frac{121}{5} - \frac{13}{3} = \frac{298}{15}$$

5. (10 pts) Evaluate

$$\lim_{n\to\infty}\sum_{k=n}^{2n}\frac{n}{k^2}$$

Ans:

$$\lim_{n\to\infty}\sum_{k=n}^{2n}\frac{n}{k^2}=\lim_{n\to\infty}\left(\frac{n}{n^2}+\sum_{k=n+1}^{2n}\frac{1}{n}\frac{1}{(\frac{k}{n})^2}\right)=\int_1^2\frac{1}{x^2}dx=\frac{1}{2}$$

6. (12 pts) Let f be a real valued function defined on $\{x\geq 0\}$ satisfying

(a): $f(0)=-1$,

(b): $f'(x)\geq 1/2$ for all $x\geq 0$.

Prove that $f(x)=0$ has <u>one and only one</u> solution on $\{x\geq 0\}$.

Ans:

Step 1:

$f(x)=0$ has <u>one</u> solution on $\{x\geq 0\}$:

Since

$$f(2)=f(0)+\int_0^2 f'(x)dx\geq -1+(2-0)\min_{0\leq x\leq 2}f'(x)\geq -1+2\times\frac{1}{2}=0$$

and $f(0)=-1<0$.

Moreover, $f'(x)$ exist for all $x\geq 0$, which implies $f(x)$ is continuous on $x\geq 0$. By Intermediate Value Theorem, there exists c between 0 and 2 such that $f(c)=0$.

Step 2:

$f(x)=0$ has <u>only one</u> solution on $\{x\geq 0\}$:

If there exist $c_1\geq 0$, $c_1\neq c$ such that $f(c_1)=0$, by Rolle's Theorem, there exist c_2 between c and c_1 such that

$$f'(c_2)=0$$

which is contradict to $f'(x)\geq\frac{1}{2}$ for all $x\geq 0$. Hence $f(x)=0$ has only one solution on $x\geq 0$.

7. (12 pts) State both parts of Fundamental Theorem of Calculus, prove that part 1 implies part 2.

Ans:

Proof of part 1 implies part 2:

Let $G(x)$ be any antiderivative of $f(x)$, which is

$$\frac{d}{dx}G(x)=f(x)$$

Since $\frac{d}{dx}\int_a^x f(t)dt = f(x) = \frac{d}{dx}G(x)$, we have

$$\int_a^x f(t)dt + c = G(x)$$

for some constant $c \in R$.

Then $G(b) - G(a) = \int_a^b f(t)dt + c - (\int_a^a f(t)dt + c) = \int_a^b f(t)dt$.

8. (12 pts) Suppose $f(x)$ satisfies $\displaystyle\int_0^{x^2} e^{-t} f(t)dt = x$ for all $x \geq 0$. Find $f(4)$ and $f'(4)$.

Ans:

Taking derivative on both sides twice gives us

$$\begin{aligned} &2xe^{-x^2} f(x^2) = 1 \\ &(2 - 4x^2)e^{-x^2} f(x^2) + 2xe^{-x^2} f'(x^2)2x = 0 \end{aligned}$$

which leads us to get

$$\begin{aligned} f(4) &= \frac{e^4}{4} \\ f'(4) &= \frac{7}{32}e^4 \end{aligned}$$