Brief solutions to Midterm Exam 1

1. (10 pts) Find $\lim_{y\to+\infty} y \sin\frac{2}{\sqrt{y}}$.

   Ans:
   $$= \lim_{y\to+\infty} \sqrt{y}\ 2\ \frac{\sqrt{y}}{2} \sin\frac{2}{\sqrt{y}} = \infty \cdot 2 \cdot 1 = \infty.$$

2. (12 pts) Give formal definition of $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and verify it using the $\varepsilon - \delta$ argument.

   Ans:

   For any $M \in R$, there exist $\delta > 0$ such that $0 < x < \delta$ implies $f(x) > M$.

   Proof: If $M > 0$, take $\delta = 1/M$. If $M \le 0$, take any $\delta > 0$ (for example $\delta = 1$) will do. Then verify (details skipped).

3. (12 pts) Find dy/dx where $y = x^{(x^x)}, x > 0$. Need not simplify your expression.

   Ans: $= x^{(x^x)} x^x (\ln x(1 + \ln x) + \frac{1}{x})$.

4. (10 pts) Find y' and y'' at $(1, -1)$ where $y(x)$ is implicitly given by $\tan^{-1}(x + y) + \sin^{-1}(x^2 + y) = 0$.

   Ans:

   $\frac{d}{dx}$ once:
   $$\frac{1 + y'}{1 + (x + y)^2} + \frac{2x + y'}{\sqrt{1 - (x^2 + y)^2}} = 0,$$
   evaluate at $x = 1, y = -1$, one gets $y'(1, -1) = -\frac{3}{2}$.

   $\frac{d}{dx}$ twice:
   $$\frac{y''(1 + (x + y)^2) - 2(x + y)(1 + y')^2}{(1 + (x + y)^2)^2} + \frac{(2 + y'')(1 - (x^2 + y)^2) + (x^2 + y)(2x + y')^2}{(1 - (x^2 + y)^2)^{\frac{3}{2}}} = 0$$
   evaluate at $x = 1, y = -1$, one gets $y''(1, -1) = -1$.

5. (12 pts) Find the smallest n such that $\frac{d^n}{dx^n}(x^{10} \sin x)|_{x=0}$ is nonzero and find this value.

   Ans: $n = 11$. $\frac{d^{11}}{dx^{11}}(x^{10} \sin x)|_{x=0} = 11!$.

6. (16 pts) True or False? If true, prove it. If false, give a counter example.

   If $|f(x) - (3x + 2)| \le |x|^{1.5}$ for all $x \in R$, then f is differentiable at $x = 0$.

   Ans: True.

   Step 1: let $x = 0$, we have $f(0) = 2$.

Step 2:

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)-(3x+2)}{x-0}+3.$$

Since $|\frac{f(x)-(3x+2)}{x-0}|\le |x|^{0.5}$, it follows from Sandwich Theorem that

$$\lim_{x\to 0}\frac{f(x)-(3x+2)}{x-0}=0.$$

Therefore

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=3,\quad \text{differentiable.}$$

7. (12 pts) Write down $L(x,x_0)$, the linear approximation of f near x_0. Find an approximate value of $\sin(\frac{\pi}{3}-0.01)$ such that the error of the approximation is smaller than 5×10^{-5}. (Hint: choose x_0 carefully)

Ans:

$L(x,x_0)=f(x_0)+f'(x_0)(x-x_0)$.

Choose $f(x)=\sin x$, $x_0=\frac{\pi}{3}$. $\sin(\frac{\pi}{3}-0.01)\approx\sin(\frac{\pi}{3})+\cos(\frac{\pi}{3})(-0.01)=\frac{\sqrt{3}}{2}-0.005$.

$$|\text{Error}|\le\frac{1}{2}|\sin c|\ 0.01^2\le 5\times 10^{-5}$$

8. (16 pts) Let f^{-1} be the inverse function of f. Evaluate $\frac{d^2}{dy^2}f^{-1}(y)$ in terms of f' and f''. Show all details.

Ans:

Let $y=f(x)$, then $x=f^{-1}(y)$ and

$$f^{-1}(y)=f^{-1}(f(x))=x,\quad \text{or } f(x)=f(f^{-1}(y))=y$$

take $\frac{d}{dx}$ twice on the first equation or $\frac{d}{dy}$ twice on the second equation both lead to

$$\frac{d^2}{dy^2}f^{-1}(y)(f'(x))^2+\frac{d}{dy}f^{-1}(y)f''(x)=0$$

which gives

$$\frac{d^2}{dy^2}f^{-1}(y)|_{y=f(x)}=-\frac{f''(x)}{(f'(x))^3},$$

or

$$\frac{d^2}{dy^2}f^{-1}(y)=-\frac{f''\left(f^{-1}(y)\right)}{\left(f'(f^{-1}(y))\right)^3},$$