

Final Exam

Jan 08, 2015

1. (14 pts) Find the solutions for $\frac{dy}{dx} = e^{x-y}$ and $x\frac{dy}{dx} + y = \sin x$, respectively.

Ans:

(1) $y = \ln(e^x + C_1)$ for some constant C_1 .

(2) $y = \frac{1}{x}(-\cos x + C_2)$ for some constant C_2 .

2. (12 pts) Find the volume and surface area of the object obtained by rotating the region $\{(x-2)^2 + y^2 \leq 1, x \geq 2\}$ around the y axis. Note the surface area consists of two parts, one generated by a half circle, the other generated by a line segment.

Ans:

Volume $= \int_{-1}^1 \pi((2 + \sqrt{1-y^2})^2 - 2^2) dy = 2\pi^2 + \frac{4}{3}\pi$.

Surface $= \int_{-1}^1 2\pi(2 + \sqrt{1-y^2})\sqrt{\frac{1}{1-y^2}} dy + 2\pi \cdot 2 \cdot 2 = 4\pi^2 + 12\pi$.

3. (8 pts) Order e^x , x^x , $(\ln x)^x$ and x^e from slowest to fastest growing rate as $x \rightarrow \infty$. Explain.

Ans:

From slowest to fastest is x^e , e^x , $(\ln x)^x$, x^x

4. (6 pts) Write down the form of partial fraction expansion for $\frac{x^7}{(1-x^4)^2}$. Need NOT find the undetermined coefficients.

Ans:

$$\frac{x^7}{(1-x^4)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x} + \frac{D}{(1+x)^2} + \frac{Ex+F}{1+x^2} + \frac{Gx+H}{(1+x^2)^2}.$$

where $A \sim H$ are undetermined coefficients.

5. (64 pts) Evaluate

(1) $\int \frac{1}{2 + \sin x} dx$ (2) $\int_1^{e^\pi} \sin(\ln x) dx$ (3) $\int \frac{1}{\sqrt{4x-x^2}} dx$ (4) $\int_1^2 \frac{1}{e^{2x} - e^{-x}} dx$

(5) $\int_0^{\pi/4} \tan^3 x \sec^3 x dx$ (6) $\int x^2 e^{-x} dx$ (7) $\int_0^1 \operatorname{sech} x dx$ (8) $\int_0^1 \frac{1}{\sqrt{1+e^x}} dx$

Ans:

(1) $= \frac{2}{\sqrt{3}} \tan^{-1}(\frac{2}{\sqrt{3}}(\tan(\frac{x}{2}) + \frac{1}{2})) + C$ for some constant C .

(2) $= \frac{e^\pi + 1}{2}$

$$(3)=\sin^{-1}\left(\frac{x-2}{2}\right)+C \text{ for some constant } C.$$

$$(4)=\frac{1}{3}\ln\left(\frac{e^2-1}{e-1}\right)-\frac{1}{6}\ln\left(\frac{e^4+e^2+1}{e^2+e+1}\right)-\frac{1}{\sqrt{3}}\left(\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(e^2+\frac{1}{2}\right)\right)-\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(e+\frac{1}{2}\right)\right)\right).$$

$$(5)=\frac{2\sqrt{2}+2}{15}.$$

$$(6)=-x^2e^{-x}-2xe^{-x}-2e^{-x}+C \text{ for some constant } C.$$

$$(7)=\tan^{-1}\left(\frac{e-e^{-1}}{2}\right).$$

$$(8)=1-2\ln(\sqrt{1+e}+1)+2\ln(\sqrt{2}+1).$$