Brief answers to Quiz 1

Oct 02, 2014

1. Give formal definition of $\lim_{x \to c} f(x) \neq L$ (just the definition, need not find ϵ or δ, etc.).

   Ans:

   There exists an $\varepsilon_0 > 0$, such that for any $\delta > 0$, $\exists x_0 \in (c - \delta, c) \cup (c, c + \delta)$ and $|f(x_0) - L| \geq \varepsilon_0$

2. Find $\lim_{\theta \to 0} \frac{\sin(\sin \theta)}{\tan 2\theta}$.

   Ans: 1/2.

3. State the Intermediate Value Theorem (Need not prove). Use it to find a c such that there is a root of "$x - 1 = \cos x$" on $(c, c + 1)$.

   Ans:

   Let $g(x) = x - 1 - \cos x$, then g is continuous on R.

   Since $g(1) = -\cos 1 < 0$, $g(2) = 1 - \cos 2 > 0$, by Intermediate Value Theorem, there exist $x_0 \in (1, 2)$ such that $g(x_0) = 0$, i.e. $x_0 - 1 = \cos x_0$. Therefore $c = 1$ works.

4. Use the $\epsilon - \delta$ argument to show that if both $f(x)$ and $g(x)$ are continuous at $x = c$, then so is $2f(x) - 3g(x)$.

   Ans:

   Since $f(x)$ and $g(x)$ are continuous at $x = c$, we have $\forall \epsilon > 0$, $\exists \delta_1 > 0, \delta_2 > 0$, such that for all x,

   $$\begin{cases} |x - c| < \delta_1 \Longrightarrow |f(x) - f(c)| < \frac{\epsilon}{5} \\ |x - c| < \delta_2 \Longrightarrow |g(x) - g(c)| < \frac{\epsilon}{5} \end{cases}$$

   Take $\delta = \min(\delta_1, \delta_2) > 0$, then for all x, $|x - c| < \delta$ implies

   $$|[2f(x) - 3g(x)] - [2f(c) - 3g(c)]| \leq 2|f(x) - f(c)| + 3|g(x) - g(c)| < \epsilon$$

   Hence $2f(x) - 3g(x)$ is also continuous at $x = c$.

5. Give formal definitions of the following limits (Just the definition, need not find δ).

   $$(a) \lim_{x \to c^-} f(x) = L \qquad (b) \lim_{x \to -\infty} f(x) = \infty$$

   Ans:

   (a)$\forall \epsilon > 0$, $\exists \delta > 0$ such that, for all x,

   $$c - \delta < x < c \Longrightarrow |f(x) - L| < \epsilon$$

   (b)$\forall B > 0$, $\exists N < 0$ such that, for all x,

   $$x < N \Longrightarrow f(x) > B$$