Brief answer to selected problems in HW05

1. Section 3.8: Problem 98. Let

$$f(x) = \ln x \Longrightarrow f'(x) = \frac{1}{x}$$

By definition, for fixed $x > 0$

$$\begin{aligned} f'(1) &= \lim_{\frac{x}{n}\to 0} \frac{f(1+\frac{x}{n})-f(1)}{\frac{x}{n}} = \lim_{n\to\infty} \frac{n}{x}(\ln(1+\frac{x}{n}) - \ln 1) \\ &= \frac{1}{x}\lim_{n\to\infty} \ln(1+\frac{x}{n})^n \end{aligned}$$

which implies

$$x = \ln(\lim_{n\to\infty}(1+\frac{x}{n})^n) \Longrightarrow \lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$$

2. Section 3.9: Problem 55. After differentiation, we find out

$$f'(x) = g'(x), \quad x > 0$$

which means

$$f(x) = g(x) + c, \quad x \geq 0$$

where c is some constant. And $x = 0$ gives us

$$-\frac{\pi}{2} = 0 + c, \quad x \geq 0$$

hence

$$f(x) = g(x) - \frac{\pi}{2}, \quad x \geq 0$$

On the other hand, we also can derive same result by following process. For $x \geq 0$, let

$$\theta = \sin^{-1}\frac{x-1}{x+1} \Longrightarrow \sin\theta = \frac{x-1}{x+1}$$

then

$$\tan(\frac{\theta+\frac{\pi}{2}}{2}) = \sqrt{\frac{1-\cos(\theta+\frac{\pi}{2})}{1+\cos(\theta+\frac{\pi}{2})}} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sqrt{x}$$

hence

$$\tan^{-1}\sqrt{x} = \frac{\theta+\frac{\pi}{2}}{2} \Longrightarrow \sin^{-1}\frac{x-1}{x+1} = 2\tan^{-1}\sqrt{x} - \frac{\pi}{2}$$

3. Hw05: Problem 3. Domain and image for csc and $\csc^{-1}$

$$\begin{aligned} \csc x &: \{x | x \in \mathbf{R}, x \neq k\pi, \forall k \in \mathbf{Z}\} \to (-\infty, -1] \cup [1, \infty) \\ \csc^{-1} x &: (-\infty, -1] \cup [1, \infty) \to \{y | -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0\} \end{aligned}$$

For $|x| > 1$, let

$$\begin{aligned} y = \csc^{-1} x &\implies \csc y = x \\ &\implies -\csc y \cot y \frac{dy}{dx} = 1 \\ &\implies \frac{dy}{dx} = \frac{-1}{\csc y \cot y} \end{aligned}$$

where

$$\cot y = \begin{cases} \sqrt{x^2 - 1} & , x > 1 \\ -\sqrt{x^2 - 1} & , x < -1 \end{cases}$$

hence

$$\frac{d \csc^{-1} x}{dx} = \frac{-1}{|x|\sqrt{x^2 - 1}}, |x| > 1$$

4. Section 3.11: Problem 64.
(a)$b_0 = f(a)$, $b_1 = f'(a)$, and $b_2 = \frac{f''(a)}{2}$
(b)Quadratic approximation of $\frac{1}{1-x}$ at $x = 0$ is $1 + x + x^2$
(d)Quadratic approximation of $\frac{1}{x}$ at $x = 1$ is $1 - (x - 1) + (x - 1)^2$
(e)Quadratic approximation of $\sqrt{1 + x}$ at $x = 0$ is $1 + \frac{1}{2}x - \frac{1}{8}x^2$
Graphs f, g, and h appear to be identical with their own quadratic approximation around the given points.

5. Hw05: Problem 5. For $\sqrt[3]{1.009}$, let

$$f(x) = (1 + x)^{\frac{1}{3}} \implies f''(x) = -\frac{2}{9}(1 + x)^{-\frac{5}{3}}$$

Then $\sqrt[3]{1.009} = f(0.009)$. Hence we use linear approximation at $x = 0$. By the error formula, we can get error bound by following process.

$$\begin{aligned} |f(0.009) - L(0.009, 0)| &\leq \frac{1}{2}(\max_{0 \leq c \leq 0.009} |f''(c)|)(0.009 - 0)^2 \\ &= \frac{1}{2} \times \frac{2}{9} \times 0.009^2 = 9 \times 10^{-6} \end{aligned}$$