Homework Assignment for Chap 13

1. Section 13.2: Problems 27, 29, 31, 34, 35, 45, 47.
2. Section 13.3: Problems 27, 35, 39, 44.
3. Section 13.4: Problems 5, 7, 8, 13, 19, 21, 29, 23, 35, 36.
4. Section 13.5: Problems 7, 11, 21, 23, 29, 31, 39, 43.
5. Section 13.6: Problem 15 (Read Example 5 for the meaning of direction of A), 21, 23, 25, 27, 37, 39, 47, 49, 51, 53, 54, 57, 58.
6. Read page 825 on definition of Linearization and think about its relation with differentiability.
7. Section 13.7: Problems 17, 35, 36, 37, 42, 45. Try to use the gradient analysis (i.e. sketch the gradient vectors to help you find the answer) in problem 37.
8. Section 13.8: Problems 7, 8, 25, 27, 29, 31. (Use method of Lagrange Multipliers only).
9. If $f(x, y)$ is continuous at $(0, 0)$ along each straight line and each parabola that passes through $(0, 0)$, i.e., if for each $a \in R$,

   $$\lim_{t\to 0} f(t, at) = \lim_{t\to 0} f(at, t) = \lim_{t\to 0} f(t, at^2) = \lim_{t\to 0} f(at^2, t) = f(0, 0).$$

   Does it mean that $\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0)$? Explain.
10. Define

    $$g(r) = \max_{\sqrt{(x-x_0)^2+(y-y_0)^2}=r} \Big| f(x, y) - f(x_0, y_0) \Big|$$

    If "$f(x, y)$ is continuous at (x_0, y_0)" equivalent to "$g(r)$ continuous at $r = 0^+$"? Explain.
11. Let $f(t) = |t|^p$. For what values of $p > 0$ is $f(t)$ continuous at $t = 0$? For what values of $p > 0$ is $f(t)$ differentiable at $t = 0$?
12. Let $f(x, y) = \sqrt{x^2 + 2y^2}$. Is $f(x, y)$ continuous at $(x, y) = (0, 0)$? Is it differentiable at $(0, 0)$?
13. Show that if $f(x, y) = o(1) \cdot |x - x_0| + o(1) \cdot |y - y_0|$ as $(x, y) \to (x_0, y_0)$ then $f(x, y) = o(1) \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2}$ and vice versa (the converse). Note that all three $o(1)$ refer to $2D$ limits as $(x, y) \to (x_0, y_0)$.
14. Find the Taylor expansion of $f(x, y, z)$ around (x_0, y_0, z_0) up to quadratic terms of x, y and z. Give an expression of the remainder term, $\frac{1}{3!}(\cdots)$.

15. True or False?

    (a) If $f(x, y)$ is differentiable at $(x, y) = (0, 0)$, then $f(x, 0)$ is differentiable at $x = 0$ and $f(0, y)$ is differentiable at $y = 0$.

    (b) If $f(x, y)$ is differentiable at $(x, y) = (0, 0)$, then for any $\theta \in [0, 2\pi)$, $f(t\cos\theta, t\sin\theta)$ is differentiable at $t = 0$.

    (c) If $f(x, 0)$ is differentiable at $x = 0$ and $f(0, y)$ is differentiable at $y = 0$, then $f(x, y)$ is differentiable at $(x, y) = (0, 0)$.

    (d) If for any $\theta \in [0, 2\pi)$, $f(t\cos\theta, t\sin\theta)$ is differentiable at $t = 0$, then $f(x, y)$ is differentiable at $(x, y) = (0, 0)$.

    (e) If for any $\theta \in [0, 2\pi)$, $f(t\cos\theta, t\sin\theta)$ is differentiable at $t = 0$, and let

    $$g(\theta) = \frac{d}{dt} f(t\cos\theta, t\sin\theta)|_{t=0}.$$

    If in addition, $g(\theta)$ is a continous fucntion of θ then $f(x, y)$ is differentiable at $(x, y) = (0, 0)$.
    Hint: If $f(x, y)$ has a tangent line at every direction θ, do these tangent lines necessarily form a tangent plane?

    (f) Consider the function

    $$f(x, y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

    Which of the assumptions in (c-e) above does f satisfy? Is f differentiable at $(0, 0)$?

16. Sketch the level curves and gradient vectors of $f(x, y) = x^2 \pm y^2$, respectively.

17. Suppose that $f_x(x, y) = 3x^2 + 2x + 2y$ and $f_y(x, y) = 2x + 2y$. Does f has a local max, local min or a saddle point at $(0, 0)$? Hint: try the gradient analysis.

18. Find the equation of plane normal to the curve

    $$\begin{cases} x^2 + 2y^2 + 3z^2 = 6 \\ x + y + z = 3 \end{cases}$$

    at $(1, 1, 1)$.

19. Suppose that the equation $F(x, y, z) = 0$ can define implicitly either $x = f(y, z)$, $y = g(z, x)$ or $z = h(x, y)$ ($F(x, y, z) = 2x + 3y - 4z - 1$ is such an example). Show that the following identity holds:

    $$f_y \cdot g_z \cdot h_x = -1.$$

    This identity is sometimes written as $x_y \cdot y_z \cdot z_x = -1$, which might be a little misleading and hard to understand.

20. The following identity

$$\int_a^b \frac{d}{dy} f(x,y)dx = \frac{d}{dy}\int_a^b f(x,y)dx \tag{1}$$

is valid provided the integrand is smooth enough. For most engineering applications including this homework problem, the integrand is smooth enough and therefore (1) holds.

Use (1) and the Chain Rule to compute

$$\frac{d}{dy}\int_1^2 \frac{\cos(xy)}{x}dx \quad \text{and} \quad \frac{d}{dy}\int_{1+y^2}^{2+\sin(y)} \frac{\cos(xy)}{x}dx$$