Homework Assignment for Week 16

1. Section 15.6: Problems 1, 15, 17, 19, 25, 27, 29, 37.
2. Section 15.7: Problems 1, 3, 7, 10, 13(a,c), 15, 16, 19.
3. This exercise is to show that Flux, Circulation and the Curl of a vector field does not depend on the coordinate you choose.

   Let x', y' be the coordinate axis obtained by rotating the x, y axis by a fixed angle θ.

   (a) Express x', y' in terms of x, y and vice versa.

   (b) Express $\frac{\partial}{\partial x'}$, $\frac{\partial}{\partial y'}$ in terms of $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and vice versa.

   (c) Let (M, N) be the components of a vector filed $\boldsymbol{F}$ in the original (x, y) coordinate. Express the components of $\boldsymbol{F}$, (M', N') in the new (x', y') coordinates in terms of M and N.

   (d) Use chain rule to verify that

   $$\frac{\partial N'}{\partial x'} - \frac{\partial M'}{\partial y'} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

   and

   $$\frac{\partial M'}{\partial x'} + \frac{\partial N'}{\partial y'} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

   (e) Express the unit vectors $\hat{x}'$, $\hat{y}'$ in terms of $\hat{x}$, $\hat{y}$ and vice versa.

   (f) Let x', y' be defined as above. In 3D, we perform the change of variable from (x, y, z) to (x', y', z) (z coordinate is unchanged). Let $(M(x, y, z), N(x, y, z), (P(x, y, z))$ be the components of a vector field $\boldsymbol{F}$ in the original (x, y, z) coordinate. Express the first two components of $\boldsymbol{F}$, (M', N') in the new (x', y', z) coordinate in terms of M and N (P remains unchanged). The same formula also works for the normal vector $\boldsymbol{n} = (n_1, n_2, n_3)$ and the tangent vector $\boldsymbol{T} = (T_1, T_2, T_3)$

   (g) Show by direct calculation that

   $$\begin{vmatrix} n_1' & n_2' & n_3 \\ \partial_{x'} & \partial_{y'} & \partial_z \\ M' & N' & P \end{vmatrix} = \begin{vmatrix} n_1 & n_2 & n_3 \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}$$

   and

   $$T_1F_1 + T_2F_2 + T_3F_3 = T_1'F_1' + T_2'F_2' + T_3F_3$$

   With the identities above, one can then perform a few successive rotations to transform a triangle lying in $\mathbb{R}^3$ into a triangle in $x - y$ plan, therefore reducing Stoke's Theorem on a triangle to Green's Theorem in $\mathbb{R}^2$. The latter can be easily verified via Fundamental Theorem of Calculus.
4. Section 15.8: Problems 1, 3, 13, 15, 17, 18, 19, 21, 23, 24.