Midterm Exam

April 28 2005, 10:10 AM. Show all details.

1. (a) Show that the series $0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \cdots + (-1)^{n-1}\frac{0.1^n}{n} + \cdots$ converges. Explain.
   (b) Find the sum of the series in (a). Need NOT explain.
   (c) Give an example of a series that converges conditionally. Explain.

2. Find a power series that converges on $(1, 3]$ and diverges elsewhere. Explain.

3. (a) State (need not prove) Taylor's Theorem for functions of one variable $f(x)$. (you can ignore the assumptions on the function and simply write down n terms approximation and the formula of the remainder term).
   (b) Use (a) to DERIVE the Taylor expansion of $f(x, y)$ up to quadratic terms. Also give a formula for the remainder term.

4. Let a parametrized curve be given by $r = \exp\theta$ in polar coordinates.
   (a) Find the slope of the curve at $\theta = \pi/2$.
   (b) Find the length of the curve on $0 \leq \theta \leq \pi$. Give your answer as a definite integral, but need NOT find the numerical value of the integral.

5. In each of the following, the coordinate in one of the 3 coordinate systems (rectangular (x, y, z), cylindrical (r, θ, z) and spherical (ρ, φ, θ)) is give. Write the coordinate in the other two coordinate systems. Restrict the answer to $r > 0, 0 \leq \theta < 2\pi$ in cylindrical system and $\rho > 0, 0 \leq \varphi \leq \pi, 0 \leq \theta < 2\pi$ in spherical system.
   (a) $(2, 0, 1)$ in cylindrical coordinate system.
   (b) $(1, \pi/2, \pi/2)$ in spherical coordinate system.

6. Suppose that the equation $F(x, y, z) = 0$ implicitly defines either $x = f(y, z)$, or $y = g(z, x)$, or $z = h(x, y)$ near a point (x_0, y_0, z_0) with $F(x_0, y_0, z_0) = 0$.
   (a) Express $\frac{\partial f}{\partial y}(y_0, z_0)$ in terms of partial derivatives of F. Explain.
   (b) Compute $\frac{\partial f}{\partial y}(y_0, z_0) \cdot \frac{\partial g}{\partial z}(z_0, x_0) \cdot \frac{\partial h}{\partial x}(x_0, y_0)$. Explain.

7. (a) Write down the definition of $\lim_{(x,y)\to(0,0)} f(x, y) = L$ in terms of ϵ and δ.
   (b) Write down the definition of '$f(x, y)$ is differentiable at $(0, 0)$'.
   (c) Find the linearization $L(x, y)$ for $f(x, y) = xy^2 + \sin(x + y)$ at $P = (1, 1)$.