Midterm Exam

Nov 11 2004, 10:10 AM

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1. (a): Write down the definition of $\lim_{x\to a} f(x) = L$, $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$ in terms of ϵ and δ respectively.

   (b): Prove using the $\epsilon - \delta$ argument that if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$, then $\lim_{x\to a} f(x) = L$.

2. Find an approximate value of $\frac{1}{\sqrt{2}}\sin(46^\circ)$ using linear approximation. Show that the error of the approximation is smaller than 5×10^{-4}.

3. Use implicit differentiation to find $y'(x)$ and $y''(x)$ at $(1,1)$ where $y(x)$ is implicitly given by $x^4 + y^2 = 2$.

4. Prove that $\tanh x = 1 - x$ has exactly one root and write down the Newton's formula for solving it. (the numerical answer is not required)

5. Graph $y = xe^{-x^2/2}$. Indicate all the asymptotes, critical points and inflection points. Find all local extrema and global extrema.

6. True or False? Give reason for your answer.

   (a) $\sin(x) = o(\sinh(x))$ as $x \to 0$,

   (b) If $f(x) = O(h(x))$ and $g(x) = O(h(x))$ as $x\to\infty$, then $f(x) = O(g(x))$ as $x\to\infty$.

7. Find the limits of the following expressions, distinguish between the cases when the limit is ∞ or $-\infty$ and the limit does not exist.

   (a) $\lim_{x\to 0^+} \dfrac{\sqrt{[x]}}{x}$

   (b) $\lim_{x\to 0^+} x^{\ln(1-x)}$

   (c) $\lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\sin^{-1} x}$

8. Let $f(x) = e^{x^2}$ defined only on $x \geq 0$.

   (a) Find the inverse function of f, indicate the domain of definition.

   (b) Determine whether the following statement is true or false. If false, point out and correct the errors:
$$\frac{df^{-1}(x)}{dx} = \frac{1}{\frac{df}{dx}} = \frac{1}{2xe^{x^2}} = \frac{1}{2x}e^{-x^2}$$