3 (a) $\ln n < n \Rightarrow (\ln n)^2 < n^2 \Rightarrow \ln(\ln n)^2 < 2\ln n \Rightarrow \frac{1}{\ln(\ln n)^2} > \frac{1}{2\ln n} > \frac{1}{2n}$ $\Rightarrow$ Since $\sum \frac{1}{2n}$ diverges, $\sum \frac{1}{\ln(\ln n)^2}$ diverges

(b) $\left\{\frac{1}{\ln(n+1)}\right\}$ decreases to zero. By alternating convergence test, $\sum \frac{(-1)^n}{\ln(n+1)}$ converges. But $\frac{1}{\ln(n+1)} > \frac{1}{n+1}$ and $\sum \frac{1}{n+1}$ diverges, so $\sum \frac{(-1)^n}{\ln(n+1)}$ converges conditionally.

(c). $\sum_{n=2}^{\infty} \ln\left(\frac{n+1}{n}\right) = \ln \prod_{n=2}^{\infty} \left(\frac{n+1}{n}\right)$. Since $\prod_{n=2}^{\infty} \left(\frac{n+1}{n}\right) \uparrow \infty$ as $n \to \infty$

$\Rightarrow \sum_{n=2}^{\infty} \ln\left(\frac{n+1}{n}\right)$ diverges. ※

5. The Taylor series of $\sin x$ at zero is "$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$"

$R_{2n+1}(x) = \frac{(-1)^n \sin^{(2n+1)}(c)}{(2n+1)!} x^{2n+1}$, where c lies between zero and x.

Fix some $x \in \mathbb{R}$ $\left|\sin^{(n)}(c)\right| \le 1 \Rightarrow |R_{2n+1}(x)| \le \frac{|x|^{2n+1}}{(2n+1)!} \to 0$, as $n \to \infty$.

$\Rightarrow \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ converges to $\sin x$ for all $x \in \mathbb{R}$. ※

6. The Taylor series of $(1+x)^{1/2}$ at zero is "$\sum_{n=0}^{\infty} \binom{1/2}{n} x^n$"

$$\lim_{n\to\infty} \left|\frac{\binom{1/2}{n+1} x^{n+1}}{\binom{1/2}{n} x^n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n\right)\cdot x^{n+1}}{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right) x^n \cdot (n+1)}\right| = \lim_{n\to\infty} \frac{2n-1}{2(n+1)} \cdot |x| < |$$

$\Rightarrow |x| < 1$, since $\lim_{n\to\infty} \frac{2n-1}{2(n+1)} = 1$. $\Rightarrow \sum_{n=0}^{\infty} \binom{1/2}{n} x^n$ converges for $|x|<1$

$\Rightarrow$ The convergence radius is 1. ※

第四題因題目有誤，故不予。

4. $1 - \frac{\pi^2}{4\cdot 2!} + \frac{\pi^4}{16\cdot 4!} + \cdots + (-1)^n \frac{\left(\frac{\pi}{2}\right)^{2n}}{(2n)!} + \cdots = \cos\frac{\pi}{2} = 0$