Chapter 6 Laplace Transform

Consider the following second order linear differential equation

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), \\ y(0) = y_0, \\ y'(0) = y_0'. \end{cases}$$

where $y : [0, \infty) \to \mathbb{R}$.
Define $\mathcal{L}\{y(t)\} : [1, \infty) \to \mathbb{R}$, by

$$\mathcal{L}\{y(t)\}(s) = \int_0^\infty e^{-st} y(t)\,\mathrm{d}t = \lim_{A\to\infty} \int_0^A e^{-st} y(t)\,\mathrm{d}t.$$

1. If $|y(t)| \le e^{at}$ for t large enough, then $\mathcal{L}\{y(t)\}(s)$ is defined for $s > a$.
2. $\mathcal{L}\{y'(t)\}(s) = \displaystyle\int_0^\infty e^{-st} y'(t)\,\mathrm{d}t,$

$$= y(t)\,e^{-st}\Big|_0^\infty - \int_0^\infty (-s)\,e^{-st} y(t)\,\mathrm{d}t,$$

$$= -y(0) + s\,\mathcal{L}\{y(t)\}(s).$$

Notation :

1. $\widehat{y}(s) = \mathcal{L}\{y(t)\}(s)$,
2. $\widehat{y'}(s) = -y(0) + s\,\widehat{y}(s)$,
3. $\widehat{y''}(s) = -y'(0) - s\,y(0) + s^2\,\widehat{y}(s)$.

Therefore, if p and q are constants, then we have

$$-y'(0) - s\,y(0) + s^2\,\widehat{y}(s) + p\,[-y(0) + s\,\widehat{y}(s)] + q\,\widehat{y}(s) = \widehat{g}(s).$$

And hence

$$\widehat{y}(s) = \frac{\widehat{g}(s) + (s+p)\,y_0 + y_0'}{s^2 + p\,s + q}.$$

Step 1 : Compute $\widehat{g}(s)$ from $g(t)$.

Step 2 : Compute $\widehat{y}(s)$ from the above algebraic equation.

Step 3 : Compute $y(t)$ from $\widehat{y}(s)$.

Example 1 : Let $y(t) = 1$, $t \geq 0$. Then

$$\widehat{1}(s) = \int_0^\infty e^{-st}\,\mathrm{d}t = \frac{1}{s}, \qquad s > 0.$$

Example 2 : Let $y(t) = e^{at}$, $t \geq 0$. Then

$$\begin{aligned}\widehat{e^{at}}(s) &= \int_0^\infty e^{-st}e^{at}\,\mathrm{d}t = \int_0^\infty e^{-(s-a)\,t}\,\mathrm{d}t \\ &= \frac{1}{s-a}, \qquad s > a.\end{aligned}$$

Example 3 : Let $y(t) = y_1(t) + y_2(t)$, $t \geq 0$. Then

$$\mathcal{L}\{c_1\,y_1(t) + c_2\,y_2(t)\}(s) = c_1\,\widehat{y_1}(s) + c_2\,\widehat{y_2}(s).$$

Example 4 : Let $y(t) = e^{iat} = \cos\,at + i\,\sin\,at$, $t \geq 0$. Then

$$\widehat{y}(s) = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2}, \qquad s > 0.$$

Therefore, we have

$$\widehat{\cos\,at}(s) = \frac{s}{s^2+a^2}, \qquad \widehat{\sin\,at}(s) = \frac{a}{s^2+a^2}.$$

Example 5 : Let $f(t) = \begin{cases} 1 & \text{if } t \geq c\,, \\ 0 & \text{if } 0 \leq t < c\,. \qquad (\,c > 0) \end{cases}$ Then

$$\widehat{f}(s) = \int_0^\infty e^{-st}\,f(t)\,\mathrm{d}t = \int_c^\infty e^{-st}\,\mathrm{d}t = \frac{e^{-sc}}{s}.$$

Example 5 : Let $g(t) = \begin{cases} f(t-c) & \text{if } t > c\,, \\ 0 & \text{if } 0 \leq t \leq c\,. \qquad (\,c \geq 0) \end{cases}$ Then

$$\widehat{g}(s) = \int_c^\infty e^{-st}\,g(t)\,\mathrm{d}t = \int_0^\infty e^{-s(\tau+c)}\,f(\tau)\,\mathrm{d}\tau = e^{-sc}\widehat{f}(s).$$

Example 6 : $\widehat{e^{-bt}\,f}(s) = \int_0^\infty e^{-st}e^{-bt}\,f(t)\,\mathrm{d}t$

$$= \int_0^\infty e^{-(s+b)t}\,f(t)\,\mathrm{d}t = \widehat{f}(s+b).$$

Convolution Operator

Define

$$\big(f*g\big)(t)=\int_0^t f(\tau)\,g(t-\tau)\,\mathrm{d}\tau=\int_0^t f(t-\tau)\,g(\tau)\,\mathrm{d}\tau.$$

Theorem : Let $h(t)=\big(f*g\big)(t)$, then $\widehat{h}(s)=\widehat{f}(s)\cdot\widehat{g}(s)$.

Proof :

$$\begin{aligned}\widehat{h}(s)&=\int_0^\infty e^{-st}\,h(t)\,\mathrm{d}t\\&=\int_0^\infty e^{-st}\int_0^t f(t-\tau)\,g(\tau)\,\mathrm{d}\tau\,\mathrm{d}t\\&=\iint_\Omega e^{-st}f(t-\tau)\,g(\tau)\,\mathrm{d}\tau\,\mathrm{d}t\\&=\int_0^\infty\left(\int_\tau^\infty e^{-st}f(t-\tau)\,\mathrm{d}t\right)g(\tau)\,\mathrm{d}\tau\\&=\int_0^\infty e^{-s\tau}g(\tau)\,\mathrm{d}\tau\,\widehat{f}(s)\\&=\widehat{f}(s)\cdot\widehat{g}(s).\end{aligned}$$

Example 7 : Let $f(t)=t$, then

$$\widehat{f}(s)=\widehat{t}(s)=\int_0^\infty e^{-st}\,t\,\mathrm{d}t=\int_0^\infty t\frac{\mathrm{d}(e^{-st})}{-s}=-\int\frac{e^{-st}}{s}\,\mathrm{d}t=\frac{1}{s^2}.$$

Example 8 : If $\widehat{h}(s)=\dfrac{a}{s^2\,(s^2+a^2)}$, for some $a\neq 0$. Then

$$\widehat{h}(s)=\frac{1}{a^2}\left(\frac{a}{s^2}-\frac{a}{s^2+a^2}\right)\qquad\text{or }\;=\frac{1}{s^2}\cdot\frac{a}{s^2+a^2}\,.$$

$$\text{Therefore } h(t)=\frac{1}{a}\cdot t-\frac{1}{a^2}\cdot\sin\,at,\qquad\text{or }\;=t*\sin\,at\,.$$

Example 9 : Let $f(t)=\begin{cases}\sin\,t & \text{if } 0\le t<\,\pi/4\,;\\ \sin\,t+\cos\left(t-\dfrac{\pi}{4}\right) & \text{if } t\ge\,\pi/4\,.\end{cases}$, then

$$f(t)=\sin\,t+\begin{cases}0 & \text{if } 0\le t<\,\pi/4\,;\\ \cos\left(t-\dfrac{\pi}{4}\right) & \text{if } t\ge\,\pi/4\,.\end{cases}$$

Therefore,

$$\widehat{f}(s) = \frac{1}{s^2+1} + \frac{s}{s^2+1} \cdot e^{-s\pi/4}.$$

Example 10 : Consider the second order initial value problem

$$\begin{cases} y'' + y = g(t), \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$$

Take " $\widehat{\ }$ " on both sides, we obtain

$$s^2\,\widehat{y} + \widehat{y} = \widehat{g}(s),$$

or

$$\widehat{y}(s) = \frac{1}{s^2+1}\,\widehat{g}(s) = \{\widehat{\sin t}\}(s) \cdot \widehat{g}(s).$$

Therefore

$$y(t) = \int_0^t \sin(t-\tau)\, g(\tau)\ \mathrm{d}\tau.$$

Impulse function

Let

$$I^{\Delta t}(t_0, t) = \begin{cases} \frac{1}{\Delta t} & \text{if } t_0 \le t \le t_0 + \Delta t, \\ 0 & \text{elsewhere.} \end{cases}$$

The formal limit is

$$\lim_{\Delta t \to 0} I^{\Delta t}(t_0, t) = \begin{cases} 0 & \text{if } t \ne t_0, \\ \infty & \text{if } t = t_0. \end{cases}$$

The point value of $\delta(t - t_0)$ doesn't make sense. It is defined as an operator

$$g(\cdot) \ \longrightarrow\ \int_0^t \delta(t-\tau)\, g(\tau)\ \mathrm{d}\tau \equiv \lim_{\Delta t \to 0} \int_0^t I^{\Delta t}(\tau, t)\, g(\tau)\ \mathrm{d}\tau = g(t).$$

That is,

$$\delta * g = g.$$

Example 1 :

$$\widehat{\delta(t-t_0)}(s) = \int_0^\infty e^{-st}\delta(t-t_0)\ \mathrm{d}t = e^{-st}\Big|_{t=t_0} = e^{-st_0}$$

Example 2 : Consider the following initial value problem

$$\begin{cases} ay'' + by' + cy = \delta(t - t_0) \\ y(t_0) = 0 \\ y'(t_0^+) = 0 \end{cases}$$

By taking the Laplace transform, we have the equation

$$as^2\widehat{y} + bs\widehat{y} + c\widehat{y} = e^{-st_0}.$$

Example 3 : Consider the following initial value problem

$$\begin{cases} az'' + bz' + cz = 0 \\ z(t_0) = 0 \\ z'(t_0^+) = 1 \end{cases}$$

By taking the Laplace transform and using the initial conditions, we obtain

$$(as^2 + bs + c)\widehat{z} - a = 0.$$