Chapter 5 Series Solutions

Consider the following second order initial value problem

$$\begin{cases} P(x)y'' + Q(x)y' + R(x)y = 0 \\ y(x_0) = y_0 \\ y'(x_0) = y'_0 \end{cases}$$

We want to find the solution of the form $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ as a convergence power series. Since $P(x_0) \neq 0$, x_0 is a regular point, we write

$$y'' + p(x)y' + q(x)y = 0, \quad \text{where } p = \frac{Q}{P}, \text{ and } q = \frac{R}{P}.$$

If $\quad p(x) = \sum p_n(x - x_0)^n$

$q(x) = \sum q_n(x - x_0)^n$

$y'_n(x) = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}$

$y''_n(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2}$

$p(x)y'(x) = p_0a_1 + (p_1a_1 + p_02a_2)(x - x_0) + \cdots$

$q(x)y'(x) = q_0a_0 + (q_1a_0 + q_0a_1)(x - x_0) + \cdots$

Comparing the Coefficients

$2a_2 + p_0a_1 + q_0a_0 = 0$

$6a_3 + p_1a_1 + 2p_0a_2 + a_0q_1 + q_0a_1 = 0$

$y(x_0) = y_0 \Rightarrow a_0 = y_0$

$y'(x_0) = y'_0 \Rightarrow a_1 = y'_0$

Thus, we can solve a_n inductively.

Theorem 1 *: If $p(x)$, $q(x)$ are analytic on $(x_0-\varepsilon, x_0+\varepsilon)$, then $\sum a_n(x-x_0)^n$ obtained by the procedure also converges on $(x_0-\varepsilon, x_0+\varepsilon)$.*

Example : Solve $\begin{cases} y''+y=0 \\ y(0)=a_0, \quad y'(0)=a_1 \end{cases}$ by series expansion.
sol :

To find $y(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$, for $x_0=0$.

Since $y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}$,

we have $\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}+\sum_{n=0}^{\infty} a_n(x-x_0)^n=0$.

Let $n-2=m$, then

$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}(x-x_0)^m+\sum_{m=0}^{\infty} a_m(x-x_0)^m=0$.

Therefore $(m+2)(m+1)a_{m+2}=-a_m, \qquad$ or $\qquad a_{m+2}=\dfrac{-a_m}{(m+2)(m+1)}$.
For example

$$a_2=\frac{-a_0}{2\,!},\ a_4=\frac{a_0}{4\,!},\ a_6=\frac{-a_0}{6\,!},\ a_3=\frac{-a_1}{3\,!},\ a_5=\frac{a_1}{5\,!},\ a_7=\frac{-a_1}{7\,!}.$$

Therefore

$$y(x)=a_0\Big(1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\frac{x^6}{6\,!}+\cdots\Big)+a_1\Big(1-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\frac{x^7}{7\,!}+\cdots\Big)$$

converges.

Remark : We can obtain an "formally" form recurrence relation. We need to determine where these power series converge.

<u>**Method 1**</u> : Please check the radius of convergence directly.

$\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\cdots, \qquad a_{2n}=\dfrac{x^{2n}}{(2n)!}$

$$\sqrt[2n]{|a_{2n}|} = \frac{|x|}{\sqrt[2n]{(2n)!}} < \frac{|x|}{R} \quad \text{where} \quad (2n)! >> R^{2n} \quad \forall R > 0$$

For n large , $\lim_{n\to\infty} \sqrt[2n]{|a_{2n}|} = 0$

$\therefore$ radius of convergence $= \infty$

Similarly, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ converges for all $x \in \mathbb{R}$.

<u>**Method 2**</u> :

$y'' + py' + qy = 0$ p, q are analytic (where $p = 0, q = 1$)

For p, q are constant, p, q are analytic.

So radius of convergence is infinite for p and q.

By Theorem, $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ converges for all $x \in \mathbb{R}$.

Example : (Airy Function)

$y'' - xy = 0$ $y(1) = a_0$ $y'(1) = a_1$

$y'' - [(x-1)+1]y = 0$

Since $(x-1)+1$ is analytic and radius of convergence is infinite,

therefore $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ converges for all $x \in \mathbb{R}$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}(x-1)^m$$

$$[(x-1)+1]y = \sum_{n=0}^{\infty} a_n(x-1)^{n+1} + \sum_{n=0}^{\infty} a_n(x-1)^n$$

$$= \sum_{m=1}^{\infty} a_{m-1}(x-1)^m + \sum_{m=0}^{\infty} a_m(x-1)^m$$

$$\Leftrightarrow \begin{array}{lcl} m=0 & , & 2a_2 = a_0 \\ m=1 & , & 6a_3 = a_0 + a_1 \\ m=2 & , & 12a_4 = a_1 + a_2 \end{array}$$

Two linearly independent solution given respectively by $y_3(x) \Leftrightarrow a_0 = 1$,

$a_1 = 0\,;\ y_4(x) \Leftrightarrow a_0 = 0\,,\ a_1 = 1$

$y_1(x)$ is a power series around $x = 0$ with $y(0) = 1$, $y'(0) = 0$.

$y_2(x)$ is a power series around $x = 0$ with $y(0) = 0$, $y'(0) = 1$.

Determine the radius of convergence

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$a_n = \begin{cases} (-1)^n \frac{1}{n!} & , \text{ if n is odd number} \\ 0 & , \text{ if n is even number} \end{cases}$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

The equality is valid when $z = z_0$.

In general, there exists $R > 0$

$$\text{s.t. } \sum_{n=0}^{\infty} a_n (z - z_0)^n \begin{cases} \text{convergent for } |z - z_0| < R \\ \text{divergent for } |z - z_0| > R \\ \text{no conclusion on } |z - z_0| = R \end{cases}$$

How to Determine R ?

$$R = \frac{1}{\overline{\lim\limits_{n\to\infty}} |a_n|^{\frac{1}{n}}}$$

Sufficient condition to determine R :

(1) If $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists, then $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \dfrac{1}{R}$

(2) If $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|}$ exists, then $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = \dfrac{1}{R}$

Remark : If $f(x) = \sum a_n (x - x_0)^n$ is valid in $(x_0 - \varepsilon, x_0 + \varepsilon)$, then $f \in C^\infty$ and

$a_0 = f(x_0),\ a_1 = f'(x_0),\ a_2 = \dfrac{f''(x_0)}{2!},$

$(1+x^2)y'' + 2xy' + 4x^2y = 0$

$\Rightarrow\ \ y'' + \dfrac{2x}{1+x^2}y' + \dfrac{4x^2}{1+x^2}y = 0$

The radius of convergence for $p(x)$, $q(x)$ around $x = \dfrac{1}{2}$ is exactly $\dfrac{\sqrt{5}}{2}$

Remark : From local existence and unique theorem

Then $y(x)$ on $-\infty < x < \infty$ for given $y(\frac{1}{2})$, $y'(\frac{1}{2})$

Example : What is the radius of convergence for given series solution of $(1+x^2)y'' + 2xy' + 4x^2y = 0$?

Sol. :

(1) around $x = 0$

$$y'' + \frac{2x}{1+x^2}y' + \frac{4x^2}{1+x^2}y = 0$$

$p(x) = \dfrac{2x}{1+x^2} = 2x(1 - x^2 + x^4)$ converges for $|x^2| < 1$

diverges for $|x^2| \geq 1$

some for $g(x) = \dfrac{4x^2}{1+x^2}$ radius of convergence $= 1$

$\therefore y(x) = \sum a_n x^n$ converges at least on $|x| < 1$.

(2) around $x = \dfrac{1}{2}$

$p(x) = \dfrac{2x}{1+x^2} = p_n(x - \dfrac{1}{2})^n$, radius of convergence $= \dfrac{\sqrt{5}}{2}$

Some for $g(x)$

$\therefore\ \ y(x) = \sum b_n(x - \dfrac{1}{2})^n$ converges at least on $|x - \dfrac{1}{2}| < \dfrac{\sqrt{5}}{2}$

Facts : (complex analytic) $\dfrac{2z}{1+z^2}$ is differentiable ($\therefore$ analytic) on $\mathbb{C}\{\pm i\}$.

If $B_\rho(\frac{1}{2}) \subseteq \mathbb{C}\{\pm i\}$ then radius of convergence for $\sum_{n=0}^{\infty} a_n(x-\frac{1}{2})^n$ is at least e

In above example, radius of convergence $= e$.

Remark : The solution exists for all $x \in \mathbb{R}$. But the power series only converges

on $(\frac{1}{2}-\frac{\sqrt{5}}{2}, \frac{1}{2}+\frac{\sqrt{5}}{2})$.

Remark : $\dfrac{1}{1+x^2}$ exists for all $x \in \mathbb{R}$. But $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} a_n(x-\frac{1}{2})^n$ only on $(\frac{1}{2}-\frac{\sqrt{5}}{2}, \frac{1}{2}+\frac{\sqrt{5}}{2})$.

$\mathbf{P}(x)y'' + \mathbf{Q}(x)y' + \mathbf{R}(x)y = 0$, $\mathbf{P}$. $\mathbf{Q}$. $\mathbf{R}$: polynomials of x.

$x = x_0$ is a ordinary point if $\lim_{x\to x_0} \dfrac{\mathbf{Q}(x)}{\mathbf{P}(x)}$ and $\lim_{x\to x_0} \dfrac{\mathbf{R}(x)}{\mathbf{P}(x)}$ exists.

Example : $\mathbf{P}(x) = x$, $\mathbf{Q}(x) = sinx$, $\mathbf{R}(x) = x^2$, $x_0 = 0$

$$\Rightarrow \begin{cases} y''(x) + \dfrac{\sin x}{x} y' + xy = 0 \\ y(0) = a_0 \quad y'(0) = a_1 \quad x \neq 0 \quad \mathbb{R} = \infty \end{cases}$$