Chapter 4 Higher order linear equation

An nth order linear differential equation is an equation of the form

$$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \cdots + p_n(t)y = g(t), \tag{1}$$

where the functions $p_1, \ldots, p_n$ and g are continuous real-valued functions on some interval $I : \alpha < t < \beta$.

The Existence and Uniqueness Theorem:

If the functions $p_1, \ldots, p_n$ and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the differential equation (1) that also satisfies the specify n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}, \tag{2}$$

where t_0 may be any point in the interval I and $y_0, y'_0, \ldots, y_0^{(n-1)}$ is any set of prescribed real constants.

The Homogeneous Equation:

We first discuss the homogeneous equation

$$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \cdots + p_n(t)y = 0. \tag{3}$$

If the functions $y_1, y_2, \ldots, y_n$ are solutions of Eq. (3), then the function

$$y(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t),$$

where $c_1, \dots, c_n$ are arbitrary constants, is also a solution of Eq. (3). In order to satisfy the initial conditions (2), we must be able to determine $c_1, \dots, c_n$ so that the equations

$$\begin{aligned} c_1 y_1(t_0) + \cdots + c_n y_n(t_0) &= y_0 \\ c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) &= y_0' \\ &\vdots \\ c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned} \tag{4}$$

are satisfied. A necessary and sufficient condition for the existence of a solution of Eqs.(4) for arbitrary values of $y_0, y_0', \dots, y_0^{(n-1)}$ is that the Wronskian

$$W(y_1, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \tag{5}$$

is not zero at $t = t_0$. It can be shown that if $y_1, \dots, y_n$ are solutions of Eq.(3), then $W(y_1, \dots, y_n)$ is either zero for every t in the interval I or else is never zero there.

The functions $f_1, \dots, f_n$ are said to be **linearly dependent** on I if there exists a set of constants $k_1, \dots, k_n$, not all zero, such that

$$k_1 f_1 + \cdots + k_2 f_2 = 0$$

for all x in I. The functions $f_1, \dots, f_n$ are said to be **linearly independent** on I if they are not linearly dependent there.

If $y_1, \dots, y_n$ are solutions of Eq.(3),then it can be shown that a necessary and sufficient condition for them to be linearly independent is that

$$W(y_1, \dots, y_n) \neq 0$$

for some t_0 in I.

Homework:

If there exist linearly independent solutions $y_1, \dots, y_n$ for the homogeneous equation, then there exists exactly one solution that also satisfies the initial conditions (2).

Homogeneous Equations with Constant Coefficients:

If $p_1(t), \dots, p_n(t)$ are constants, then $y_1, \dots, y_n$ are given by $p(t)e^{r_j t}$, where r_j are solutions of

$$Q(r) = r^n + p_1 r^{n-1} + \cdots + p_n = 0.$$

If n distinct roots

1. real roots $\rightarrow e^{rt}$

2. complex roots $e^{\alpha \pm i\beta} \rightarrow e^{\alpha t} \cos \beta t$, $e^{\alpha t} \sin \beta t$

Recall $n = 2$: r is a double root $\Leftrightarrow Q(r) = Q'(r) = 0$. Then e^{rt}, te^{rt} are solutions.

general n: r is a root of multiplicity $m \Leftrightarrow Q(r) = Q'(r) = \cdots = Q^{m-1}(r) = 0$.

claim: $e^{rt}, te^{rt}, \cdots, t^{m-1}e^{rt}$ are solutions.

Note that $(t^k e^{rt})' = r(t^k e^{rt}) + kt^{k-1}e^{rt}$.

Consider $p(t)e^{rt} = y(t)$, then

$$\begin{aligned}
(pe^{rt})' &= p'e^{rt} + rpe^{rt} \\
(pe^{rt})'' &= p''e^{rt} + 2rp'e^{rt} + r^2pe^{rt} \\
&\vdots \\
(pe^{rt})^{(n)} &= \tbinom{n}{0}p^{(n)}e^{rt} + \tbinom{n}{1}rp^{(n-1)}e^{rt} + \cdots + \tbinom{n}{n}r^npe^{rt}
\end{aligned}$$

Since

$$\begin{aligned}
a_0\tbinom{n}{n}r^npe^{rt} + a_1\tbinom{n-1}{n-1}r^{n-1}pe^{rt} + \cdots + a_npe^{rt} &= Q(r)pe^{rt} \\
a_0\tbinom{n}{n-1}r^{n-1}p'e^{rt} + a_1\tbinom{n-1}{n-2}r^{n-2}p'e^{rt} + \cdots &= Q'(r)p'e^{rt} \\
a_0\tbinom{n}{n-2}r^{n-2}p''e^{rt} + \cdots &= \tfrac{1}{2!}Q''(r)p''e^{rt} \\
&\vdots \\
a_0\tbinom{n}{0}p^{(n)}e^{rt} + a_1\tbinom{n-1}{0}p^{(n-1)}e^{rt} + \cdots &= \tfrac{1}{n!}Q^{(n)}(r)p^{(n)}e^{rt}
\end{aligned}$$

and

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = 0,$$

therefore

$$Q(r)p + Q'(r)p' + \cdots + \frac{1}{2!}Q''(r)p'' + \cdots + \frac{1}{n!}Q^{(n)}(r)p^{(n)} = 0.$$

Since

$$Q(r) = Q'(r) = \cdots = Q^{(m-1)} = 0;$$

we obtain

$$\frac{1}{m!}Q^{(m)}(r)p^{(m)} + \cdots + \frac{1}{n!}Q^{(n)}(r)p^{(n)} = 0.$$

If $p(t) = t^k$, where $0 \le k \le (m-1), k \in \mathbb{N}$,

then

$$p^{(m)} = p^{(m+1)} = \cdots = p^{(n)} = 0.$$

Hence $e^{rt}, te^{rt}, \cdots, t^{m-1}e^{rt}$ are solutions.

Inhomogeneous case:

- Undertemined coefficient

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = g(t), \quad (a_0 \neq 0)$$

- $g(t)$ is a polynomial of degree m, say $g(t) = c_0 t^m + \cdots + c_m$.

  Try

$$y(t) = b_0 t^m + b_1 t^{m-1} + \cdots + b_{m-2}t^2 + b_{m-1}t + b_m,$$

  Then

$$\begin{aligned} y'(t) &= m\, b_0 t^{m-1} + (m-1)b_1 t^{m-2} + \cdots + 2b_{m-2}t + b_{m-1}, \\ y''(t) &= m(m-1)\, b_0 t^{m-2} + (m-1)(m-2)b_1 t^{m-3} + \cdots + 2b_{m-2}, \\ &\vdots \end{aligned}$$

And hence

$$\begin{aligned}
&a_n b_0 = c_0,\\
&a_n b_1 + m\, a_{n-1} b_0 = c_1,\\
&a_n b_2 + (m-1)\, a_{n-2} b_1 + m(m-1) a_{n-2} b_0 = c_2,\\
&\vdots
\end{aligned}$$

We can solve b_j inductively.

- If $t(t) = e^{rt}$, for some $r \in \mathbb{R}$ and

$$Q(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_n \neq 0,$$

then

$$y(t) = \frac{e^{rt}}{Q(r)}.$$

- $g(t) = \cos \sigma t$ i.e. $\mathrm{Re}(e^{i\sigma t})$, and $Q(i\sigma) \neq 0$, then

$$y(t) = \mathrm{Re}\Big(\frac{e^{i\sigma t}}{Q(i\sigma)}\Big).$$

- $g(t) = \sin \sigma t$ i.e. $\mathrm{Im}(e^{i\sigma t})$, and $Q(i\sigma) \neq 0$, then

$$y(t) = \mathrm{Im}\Big(\frac{e^{i\sigma t}}{Q(i\sigma)}\Big).$$

Homework:

If $Q(r) = Q'(r) = \cdots = Q^{m-1}(r) = 0, \quad Q^m(r) \neq 0$

find a solution $y(t)$

Hint: try $y(t) = q(t)e^{rt}$

Variation of parameters

Let $y_1, \ldots, y_n$ be n linearly independent to the homegeneous equation

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0. \tag{6}$$

Now, we want to find a soluiton to the inhomegeneous equation

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = g(t). \tag{7}$$

Try

$$y(t) = u_1(t)y_1(t) + \cdots + u_n(t)y_n(t).$$

First, we impose $n-1$ additional conditions on $u_1, \ldots, u_n(t)$ as following:

Since

$$y'(t) = u_1'(t)y_1(t) + \cdots + u_n'(t)y_n(t) + (u_1 y_1' + \cdots + u_n y_n')$$

Impose $u_1'(t)y_1(t) + \cdots + u_n'(t)y_n(t) = 0$. Again

$$y''(t) = u_1'(t)y_1'(t) + \cdots + u_n'(t)y_n'(t) + (u_1 y_1'' + \cdots + u_n y_n'')$$

Impose $u_1'(t)y_1'(t) + \cdots + u_n'(t)y_n'(t) = 0$. Continuing in this way

$$\vdots$$

And

$$y^{(n)}(t) = u_1'(t)y_1^{(n-1)}(t) + \cdots + u_n'(t)y_n^{(n-1)}(t) + (u_1 y_1^n + \cdots + u_n y_n^n)$$

By (6) and (7), we obtain the following system

$$\begin{cases} u_1'y_1 + \cdots + u_n'y_n = 0, \\ u_1'y_1' + \cdots + u_n'y_n' = 0, \\ \qquad\qquad \vdots \\ u_1'y_1^{(n-1)} + \cdots + u_n'y_n^{(n-1)} = g(t). \end{cases}$$

And hence $u_j' = \dfrac{w_j}{w}$, where

$$w = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \vdots \\ y_n^{n-1} & \cdots & y_n^{n-1} \end{vmatrix} \text{,and } w_j = \begin{vmatrix} y_1 & \cdots & 0 & \cdots & y_n \\ y_1' & \cdots & 0 & \cdots & y_n' \\ \vdots & & \vdots & & \vdots \\ y_n^{n-1} & \cdots & g & \cdots & y_n^{n-1} \end{vmatrix}.$$

Homework:

- Let $u_j(t) = \displaystyle\int_{t_0}^{t} \frac{w_j(s)}{w(s)}\, ds$. What are the initial conditions of $y(t)$?

- What is the Duhamel's formula in this case?