Chapter 3 Second Order Differential Equation

Linear Superposition Priciple

$y'' + p(t)y' + q(t)y = 0$ (linear ODE)

$y_1(t) \cdot y_2(t)$ are solutions :

$\Rightarrow c_1 y_1(t) + c_2 y_2(t)$ is also a solution

$ay'' + by' + cy = 0 \quad a\,, b\,, c \in \mathbb{R}$

Try e^{rt} , r to be determined

$\Rightarrow (ar^2 + br + c)\, e^{rt} = 0$

If r is a solution of $ar^2 + br + c = 0$, then e^{rt} is a solution of

$$ay'' + by' + cy = 0.$$

3 cases

case (1) : $r = r_1\,, r_2 \qquad r_1\,, r_2 \in \mathbb{R}$, and $r_1 \neq r_2$.

$\exists$ at least 2 solutions $e^{r_1 t}\,, e^{r_2 t}$ & $c_1 e^{r_1 t} + c_2 e^{r_2 t}$

case (2) : $r = \alpha + \beta i \qquad \alpha\,, \beta \in \mathbb{R}$

$\Rightarrow e^{(\alpha+i\beta)t}\,, e^{(\alpha-i\beta)t}$ are solutions of $ay'' + by' + cy = 0$

$c_1\, e^{(\alpha+i\beta)t} + c_2\, e^{(\alpha-i\beta)t}$

$= (c_1 + c_2)\, e^{\alpha t}(\cos\beta t) + i(c_1 - c_2)\, e^{\alpha t} \sin\beta t$

$= \widetilde{c_1}\, e^{\alpha t} \cos\beta t + i\, \widetilde{c_2}\, e^{\alpha t} \sin\beta t$

$e^{(\alpha+i\beta)t} = e^{\alpha t} \cdot e^{i\beta t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$

$e^{(\alpha-i\beta)t} = e^{\alpha t} \cdot e^{-i\beta t} = e^{\alpha t}(\cos\beta t - i\sin\beta t)$

case (3) : double roots r_1 , r_2

e^{rt} and te^{rt} are 2 solutions. $(b^2 = 4ac)$

For (3), i.e. $r = -b/2a$, e^{rt} is a solution

Try $p(t)e^{rt}$ as follow:

$$(p(t)e^{rt})' = p'e^{rt} + rpe^{rt}$$
$$(p(t)e^{rt})'' = p''e^{rt} + 2rp'e^{rt} + r^2pe^{rt}$$

Pluging into $ay'' + by' + cy = 0$, we get

$$a(p'' + 2p'r + pr^2) + b(p' + pr) + cp = 0$$
$$ap'' + (2ar + b)p' + (ar^2 + br + c)p = 0$$

Hence $ap'' = 0$ and we may choose $p(t) = t$.

Question : all 3 cases $\exists\, y_1\,(t)\,, y_2\,(t)$

$y_1 \neq$ constant $y_2\,(t)$

Are there solutions not of the form $c_1y_1(t) + c_2y_2(t)$?

Ans

$$\begin{cases} ay'' + by' + cy = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_0 \end{cases}$$

has exactly one solution.

Eg e^{r_1t} , e^{r_2t} , $r_1 \neq r_2 \in \mathbb{R}$

If another solution $y_*(t)$

Can we find c_1 , c_2

$$\begin{cases} c_1y_1(0) + c_2y_2(0) = y_*(0) \\ c_1y_1'(0) + c_2y_2'(0) = y_*'(0) \end{cases}$$

If we can find c_1, c_2

$$\Rightarrow \begin{cases} c_1 y_1(t) + c_2 y_2(t) & \text{have the same } y(0) \text{ and } y'(0) \\ y_*(t) \end{cases}$$

From uniqueness $\Rightarrow y_*(t) = c_1 y_1(t) + c_2 y_2(t)$

$$\begin{array}{ll} y_1(0) = 1 & y_2(0) = 1 \\ y_1'(0) = r_1 & y_2'(0) = r_2 + r_1 \quad \Rightarrow c_1 , c_2 \text{ exists}. \end{array}$$

2nd Order Linear ODE

(1) **Homogeneous ODE**

$$y'' + p(t)\, y' + q(t)\, y = 0$$

Existence of linearly independent solution

Consider $ar^2 + br + c = 0$

3 cases

case (1) : 2 real roots $r_1 \neq r_2$

$\Rightarrow c_1 e^{r_1 t} + c_2 e^{r_2 t}$ the solutions

case (2) : multiple real roots r , r

$\Rightarrow c_1 e^{rt} + c_2 t e^{rt} \quad \forall\, c_1 , c_2$

case (3) : 2 complex conjugate roots

$\alpha \pm i\,\beta \qquad \alpha , \beta \in \mathbb{R} \quad \beta \neq 0$

$\Rightarrow c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t \qquad \forall c_1 , c_2$.

Given $y_1(t)$, $y_2(t)$ 2 linear indep solution

homogeneous 2nd order ODE

How to construct solution of

$$(\star\star)\begin{cases} ay'' + by' + cy = 0 \\ y\,(0) = y_0 \\ y'\,(0) = y_0' \end{cases}$$

Ans : Find $c_1\,,\, c_2 \in \mathbb{R}$ s.t.

$$(\star)\begin{cases} c_1 y_1(0) + c_2 y_2(0) = y_0 \\ c_1 y_1'(0) + c_2 y_2'(0) = y_0' \end{cases}$$

$\Rightarrow c_1 y_1(t) + c_2 y_2(t)$ is the solution.

$(\star)$ has a solution

given $y_0\,,\, y_0'$, can always find unique $c_1\,,\, c_2$.

iff $\begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} \neq 0$

i.e. $\begin{pmatrix} y_1(t) \\ y_1'(t) \end{pmatrix} \begin{pmatrix} y_2(t) \\ y_2'(t) \end{pmatrix}$ are linear indep

in $\mathbb{R}^2$ at $t = 0$.

Definition : $y_1(t)$, $y_2(t)$ are linearly dependent

iff $\exists\, c_1\,, c_2 \in \mathbb{R}$ s.t. $c_1\, y_1(t) + c_2\, y_2(t) \equiv 0$ for all $t \in (\alpha\,,\beta)$.

Linear independent $\Leftrightarrow$ NOT linear dependent.

Proposition : $y_1(t) = c\, y_2(t)$

$$\Rightarrow \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = 0$$

Theorem : $p\,, q$ continuous on $(\alpha\,,\beta)$

$y_1\,,y_2$ are solutions of $y'' + p\,(t)y' + q(t)y = 0$

$$\begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = 0 \;\Leftrightarrow\; \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = 0$$

$\forall t \in (\alpha\,,\beta) \quad 0 \in (\alpha\,,\beta)$

Define $w(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$

$$\Rightarrow w'(t) = \begin{vmatrix} y_1'(t) & y_2'(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} + \begin{vmatrix} y_1(t) & y_2(t) \\ y_1''(t) & y_2''(t) \end{vmatrix}$$

$$\Rightarrow w'(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ -py_1' - qy_1 & -py_2' - qy_2 \end{vmatrix} = -p\,(t)\,w(t)$$

$$\therefore w(t) = w(0)\exp\,(\,-\int_0^t p\,(s)\,ds\,)\,.$$

$w(t) = 0 \Leftrightarrow w(0) = 0$

Corollary : If $y_1(t)\,,\,y_2(t)$ are two solutions of $ay'' + by' + cy = 0$ with

$$\begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} \neq 0$$

Then the solution to $(\star\star)$ is uniquely given by $(\star)$.

Proof : The existence has been done, $\because c_1\,y_1\,(t) + c_2\,y_2\,(t)$ satisfies $(\star\star)$

And the uniqueness is equivalent to

$$(\star\star\star)\begin{cases} ay'' + by' + cy = 0 \\ y(0) = y'(0) = 0 \end{cases} \quad \text{implies } y(t) = 0.$$

Because if $y_A(t)$ and $y_B(t)$ both solve $(\star\star)$, then $y_A - y_B$ solve $(\star\star\star)$

Why is $(\star\star\star)$ true?

Consider $w_1(t) = \begin{vmatrix} y_1(t) & y_1(t) \\ y'(t) & y'(t) \end{vmatrix}$ and $w_2(t) = \begin{vmatrix} y_2(t) & y_2(t) \\ y'(t) & y'(t) \end{vmatrix}$

Since $y(0) = y'(0) = 0 \Rightarrow w_1(0) = w_2(0) = 0$

$$\Rightarrow w_1(0) = w_2(0) = 0$$

i.e. $\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$ is linearly dependent with both

$\begin{pmatrix} y_1(t) \\ y_1'(t) \end{pmatrix}$ and $\begin{pmatrix} y_2(t) \\ y_2'(t) \end{pmatrix}$. Therefore, $y(t) = 0\,,\ y'(t) = 0$

Example

$$\begin{cases} y'' + k^2 y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases} \quad \text{— (1)} \quad \text{and} \quad \begin{cases} y'' + k^2 y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases} \quad \text{— (2)}$$

(1) : $y(t) = \cos kt$

(2) : $y(t) = \frac{1}{k} \sin kt$

In general

$$\begin{cases} y'' + k^2 y = 0 \\ y(0) = A \\ y'(0) = B \end{cases}$$

$\Rightarrow y(t) = A \cos kt + B \frac{1}{k} \sin kt$

Remark

"The existence of two linear indep. $\Rightarrow$ unique" is also valid for $y'' + p(t)y + q(t)y = 0$.

But "existence of Two linear indep. solutions" was not proved in class.

Example : Solve the following initial value problem

$$\begin{cases} y'' - k^2 y = 0 \\ y(0) = A \\ y'(0) = B \end{cases}$$

First note that $y'' - k^2 y = 0$ has a family of solutions $y = c_1 e^{kt} + c_2 e^{-kt}$. According to the initial condition, we have

$$\begin{aligned} A &= y(0) = c_1 + c_2 \\ B &= y'(0) = c_1 k - c_2 k \end{aligned}$$

Hence, $c_1 = (Ak + B)/2k$ and $c_2 = (Ak - B)/2k$.

Inhomogeneous 2nd Order ODE

Suppose $y_1(t)$, $y_2(t)$ are two linear indep solutions in the case of $g(t) = 0$. Then

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(0) = y_0 \,, y'(0) = y_0' \end{cases}$$

has a solution of the form

$$y = c_1 y_1(t) + c_2 y_2(t) + Y(t),$$

where $c_1 y_1 + c_2 y_2$ is the solution to the homogeneous initial value problem

$$\begin{cases} y'' + py' + qy = 0 \\ y(0) = y_0, \qquad y'(0) = y_0' \end{cases}$$

and $Y(t)$ is the solution to the equation

$$\begin{cases} y'' + py' + qy = g(t) \\ y(0) = 0, \qquad y'(0) = 0 \end{cases}$$

Remark Given (1) $\Rightarrow$ solution to (2) is unique , same proof as before.

Remark If given $Y(t)$

$$\begin{cases} Y'' + pY' + qY = g \\ Y(0) = A \qquad Y'(0) = B \end{cases}$$

$\Rightarrow$ solution to (2) is given by $c_1 y_1(t) + c_2 y_2(t) + Y(t)$.

Inhomogeneous 2nd Order Linear ODE

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) ----(1) \\ y(t_0) = y_0 \qquad y'(t_0) = y_0' \end{cases}$$

For homogeneous ODE

$y'' + p(t)y' + q(t)y = 0 ----(2)$

$\exists\, y_1(t)\,, y_2(t)$ is linear independent solutions

i.e. $w(y_1, y_2) \neq 0$ $(\forall t)$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} (t)$$

$w' + pw = 0$

such that all solutions of (2) is of the $c_1 y_1(t) + c_2\, y_2(t)$

$y_1(t)\,, y_2(t)$ can be explicitly found for the constant coefficient case. For the general case, we only know the existence of $y_1(t)\,,\ y_2(t)$ from theorem of the solution to (1) is unique.

We can decompose the solution as $y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$ where $y_1\,,\ y_2$ are linear independent solutions of (2), $Y(t)$ is any solution to $y'' + py' + qy = g\,.$ $c_1\,,$ c_2 are chosen so that $y(t_0) = y_0\,,\ y'(t_0) = y_0'\,.$

Next, we introduce some methods that picks a $Y(t)$ (special solution) in terms of $y_1(t)$ and $y_2(t)\,.$

(a) guess (method of undetermined coefficient)

(b) variation of parameter .

Variation of Parameter

Try $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ with $u_1\,,\ u_2$ to be determined. There are some freedom in choosing u_1, u_2. Put them into $y'' + p(t)y' + q(t)y = g(t)$, we obtain

$$\begin{aligned} y'' + py' + qy &= u_1''y_1 + u_2''y_2 + 2u_1'y_1' + 2u_2'y_2' + u_1y_1'' + u_2y_2'' \\ &\quad + p(u_1'y_1 + u_2'y_2 + u_1y_1' + u_2y_2') + q(u_1y_1 + u_2y_2) \\ &= g(t) \end{aligned}$$

We impose $u_1'y_1 + u_2'y_2 = 0 - - - -(1)$

$\Rightarrow u_1'y_1' + u_2'y_2' = g(t) - - - -(2)$

$$\Rightarrow \begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g(t) \end{cases}$$

solve for u_1' , u_2'

$$u_1'(t) = \frac{-y_2(t)g(t)}{w(y_1\,, y_2)(t)} \quad \text{and} \quad u_2'(t) = \frac{y_1(t)g(t)}{w(y_1\,, y_2)(t)}$$

$$u_1(t) = \int_{t_0}^{t} \frac{-y_2 g}{w}\, ds \quad \text{and} \quad u_2(t) = \int_{t_0}^{t} \frac{-y_1 g}{w}\, ds$$

$\Rightarrow Y(t_0) = u_1(t_0)y_1(t_0) + u_2(t_0)y_2(t_0) = 0$

$Y'(t_0) = (u_1'y_1 + u_2'y_2)(t_0) + (u_1y_1' + u_2y_2')(t_0) = 0$

$$\Rightarrow \begin{cases} Y'' + pY' + qY = g(t) \\ Y(t_0) = 0 \qquad Y'(t_0) = 0 \end{cases}$$

Find a particular solution of $y'' + 2y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t$

Consider $y'' + 3y' + 4y$

$y_1(t)$, $y_2(t)$ are the solutions of $y'' - 3y' - 4y = 0$

$\Rightarrow y_1(t) = e^{4t}$, $y_2(t) + e^{-t}$

By the linear superposition, it suffices to find the solutions of

$$y'' - 3y' - 4y = \begin{cases} 3e^{2t} \\ 2\sin t \\ -8e^t \cos 2t \end{cases} .$$

(1) $y'' - 3y' - 4y = 3e^{2t}$

$2 \neq r_1\,, r_2$

$(e^{2t})'' - 3(e^{2t})' - 4(e^{2t}) = 4e^{2t} - 6e^{2t} - 4e^{2t} \neq 0$

Try $y = Ae^{2t}$

$\Rightarrow 4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$

$-6A = 3 \Rightarrow A = -1/2$

$\Rightarrow y(t) = -1/2e^{2t}$

(2) $y'' - 3y' - 4y = 2\sin t = 2\,\mathrm{Im}\,e^{it}$

Try $y = A\cos t + B\sin t$ (Equivalent to try $(C + Di)e^{it}$) as follow :

$y'' - 3y' - 4y$

$= -B\sin t - A\cos t + 3A\sin t - 3B\cos t - 4A\cos t - 4B\sin t$

$= 2\sin t$

$$\begin{cases} -B + 3A - 4B = 2 \\ -A - 3B - 4A = 0 \end{cases} \Rightarrow \begin{cases} 3A - 5B = 2 \\ 5A + 3B = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{6}{34} \\ B = \frac{-10}{34} \end{cases}$$

$$\begin{pmatrix} -A \\ -B \end{pmatrix} - 3\begin{pmatrix} B \\ -A \end{pmatrix} - 4\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$(C + Di)(e^{it})'' - 3(e^{i}t)' - 4(e^{it})$

$= (C + Di)(-5 - 3i)e^{it}$

$= (-5C - 3Ci - 5Di + 3D)e^{it}$

$= (-5C + 3D + i(-3i - 5D))e^{it}$

$= 2\,\mathrm{Im}\,e^{it}$

$$\Rightarrow \begin{cases} -5C + 3D = 2 \\ -3C - 5D = 0 \end{cases} \Rightarrow \begin{cases} C = -\frac{10}{34} = \frac{-5}{17} \\ D = \frac{6}{34} = \frac{3}{17} \end{cases}$$

(3) $y'' - 3y' - 4y = -8\, e^t \cos 2t = -8 \mathrm{Re}\, e^{(1+2i)t}$

$\lambda \left((e^{(1+2i)t})'' - 3\,(e^{(1+2i)t})' - 4\,(e^{(1+2i)t})\, \right)$

$= \lambda\,(-2i - 10)\, e^{(1+2i)t} = -8\, e^{(1+2i)t}$

$\Rightarrow \lambda = \frac{-8}{-2i-10}$

$\Rightarrow$ take $\quad y = \mathrm{Re}\,(\lambda\, e^{(1+2i)t})$

Example : To find a particular solution of $y'' + 4y = 3\cos(2t)$, $\big(= 3\,\mathrm{Re}\, e^{2it}\big)$.

First we solve $r^2 + 4 = 0$, and obtain $r = \pm 2i$.

Next, try $\lambda t\, e^{2it}$ as follow :

Take $\mathrm{Re}\,(\lambda\, t e^{2it})$

$y'' - 4y' + 4y = e^{2t}$

Try $\quad A\, t^2\, e^{2t}$

Consider :

$$\begin{cases} ay'' + by' + cy = ke^{qt} \\ aq^2 + bq + c = 0 \end{cases}$$

Try $p(t)e^{qt}$

$2aq + b \neq 0 \Leftrightarrow q = r_1 \neq r_2$

$2aq + b = 0 \Leftrightarrow r_1 = r_2$

$(p\,(t)\, e^{qt})'' + (p\,(t)\, e^{qt})' + (p\,(t)\, e^{qt})$

$$a \begin{pmatrix} p'' \\ +2p'q \\ +p\, q^2 \end{pmatrix} e^{qt} + b \begin{pmatrix} p' \\ p\, q \end{pmatrix} e^{qt} + c\,\big(\ p\ \big)\, e^{qt}$$

$= p''ae^{qt} + p'(2aq+b)e^{qt} + p(aq^2+bq+c)e^{qt}$

$= ke^{qt}$

There are two possibility :

1. If $2aq + b = 0$, then we let $p'' = k/a$.

2. If $2aq + b \neq 0$, then we let $p'' = 0$ and $p' = k/(2aq+b)$.

Example : To solve the differential equation $y'' + 3y' + 2y = t^k$, we may try the polynomial $y(t) = a_k t^k + a_{k-1}t^{k-1} + \cdots + a_0$. Then

$$\begin{aligned}
&a_k k(k-1)t^{k-2} + a_{k-1}(k-1)(k-2)t^{k-3} + \cdots \\
&+3\Big(a_k k t^{k-1} + a_{k-1}(k-1)t^{k-2} + a_{k-2}(k-2)t^{k-3} + \cdots\Big) \\
&+2\Big(a_k t^k + a_{k-1}t^{k-1} + a_{k-2}t^{k-2} + \cdots\Big) = t^k.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&2a_k = 1 \\
&3a_k k + 2a_{k-1} = 0 \\
&a_k k(k-1) + 3a_{k-1}(k-1) + 2a_{k-2} = 0 \\
&\vdots
\end{aligned}$$

Question : What can we do in the case of $y'' + 3y' = t^k + \cdots$?

Answer : Try the polynomial $y(t) = a_{k+1}\, t^{\,k+1} + a_k\, t^k + \cdots$.

Remark : To solve $y'' = t^k + \cdots$, we integrate on both sides directly.

Undamped Osoillations

Consider the second order differential equation

$$mu'' + ku = 0,$$

for some positive constants m, k.

$(F = ma)$

$u'' + \frac{k}{m} u = 0$

$u = A\cos\left(\sqrt{\frac{k}{m}}\, t\right) + B\sin\left(\sqrt{\frac{k}{m}}\, t\right) = \sqrt{A^2+B^2}\cos\left(\sqrt{\frac{k}{m}}\, t + \delta_0\right).$

$T = \dfrac{2\pi}{\sqrt{k/m}}$

Damped Oscillations

$mu'' + \nu u' + ku = 0, \quad \nu > 0$ (fixed)

The solution are given by $c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$r_1\,,\, r_2\,:\; mr^2 + \nu r + k = 0$

$r_1\,, r_2 = \dfrac{-\nu \pm \sqrt{\nu^2 - 4mk}}{2m}$

There are 3 cases to be discussed :

1. If $\nu^2 - 4mk > 0$, then $r_1 < 0$, $r_2 < 0$, and $r_1 \neq r_2$.

2. If $\nu^2 - 4mk = 0$, then $r_1 = r_2 = \dfrac{-\nu}{2m} < 0$.

3. If $\nu^2 - 4mk < 0$, then $\operatorname{Re} r_1 = \operatorname{Re} r_2 = \dfrac{-\nu}{2m} < 0$.

Since $u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, i.e.

$$u = e^{-\nu t/2m}\left(c_1\cos(\frac{\sqrt{4mk-\nu^2}}{2m}t) + c_2\sin(\frac{\sqrt{4mk-\nu^2}}{2m}t)\right),$$

we have $\lim_{t\to\infty} u(t) = 0$ for any initial data $u(0)$, $u'(0)$.

REMARK :

1. In the case of $r_1 \neq r_2 < 0$, say $r_1 < r_2 < 0$, we have $u = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

   $c_1\,, c_2 = \,?$

   $$\begin{cases} u(0) = c_1 + c_2 \\ u'(0) = r_1 c_1 + r_2 c_2 \end{cases}$$

   Therefore, $c_1 = \dfrac{u'(0) - r_2 u(0)}{r_1 - r_2}$, and $c_2 = \dfrac{u'(0) - r_1 u(0)}{r_2 - r_1}$.

2. In the case of $r_1 = r_2 < 0$, we have $u(t) = c_1 e^{rt} + c_2 t e^{rt}$.

Undamped Oscillations

$mu'' + ku = F_0 \cos \omega t$

Let $\omega_0 = \sqrt{k/m}$

$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{k - m\omega^2} \cos \omega t$ $(\omega^2 \neq \omega_0^2 \Rightarrow k - m\omega^2 \neq 0)$

$c_1\,, c_2 = \,?$

$$\begin{cases} c_1 + \frac{F_0}{k-m\omega^2} = u(0) \\ \omega_0 c_2 = u'(0) \end{cases}$$

$u = \sqrt{c_1^2 + c_2^2}\, \cos\left(\omega_0 t + \delta\right) + \frac{F_0}{k-m\omega^2}\, \cos \omega t$

As $\omega \sim \omega_0$

$$\begin{cases} \omega_0 = \dfrac{\omega + \omega_0}{2} - \dfrac{\omega - \omega_0}{2} \\ \omega = \dfrac{\omega + \omega_0}{2} + \dfrac{\omega - \omega_0}{2} \end{cases} \quad \text{implies} \quad \begin{cases} \dfrac{\omega + \omega_0}{2} \sim \omega_0 \\ \dfrac{\omega - \omega_0}{2} \sim 0 \end{cases}$$

For simplicity, suppose $u'(0) = 0$ $(c_2 = 0) \Rightarrow \delta = 0$

$u = c_1 \cos \omega_0 t + \frac{F_0}{k-m\omega^2} \cos \omega t$

Let $A = \frac{\omega + \omega_0}{2} t$, and $B = \frac{\omega - \omega_0}{2} t$

$\Rightarrow u = c_1(\cos A \cos B + \sin A \sin B) + \frac{F_0}{k-m\omega^2}(\cos A \cos B - \sin A \sin B)$

$$= \begin{cases} \frac{2F_0}{k-m\omega^2} \cos(\frac{\omega+\omega_0}{2}t) \cos(\frac{\omega-\omega_0}{2}t) & c_1 = \frac{F_0}{k-m\omega^2} \\ \frac{-2F_0}{k-m\omega^2} \sin(\frac{\omega+\omega_0}{2}t) \sin(\frac{\omega-\omega_0}{2}t) & c_1 = \frac{-F_0}{k-m\omega^2} \end{cases}$$

Recall : $mu'' + ku = F_0 \cos \omega t$

$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2-\omega^2)} \cos \omega t$

$\omega_0^2 = \frac{k}{m}$

If $u(0) = u'(0) = 0$, $c_2 = 0$, $c_1 = \frac{-F_0}{m(\omega_0^2-\omega^2)}$

then

$$\begin{aligned} u(t) &= \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t) \\ &= \frac{-2F_0}{m(\omega_0^2 - \omega^2)} \cdot \sin(\tfrac{\omega+\omega_0}{2}t) \cdot \sin(\tfrac{\omega-\omega_0}{2}t). \end{aligned}$$

In general,

$$\begin{aligned} u(t) &= A\cos(\omega_0 t + \delta) + B\cos(\omega t) \\ &= A(\cos\alpha\cos\beta + \sin\alpha\sin\beta) + B(\cos\alpha\cos\beta - \sin\alpha\sin\beta) \\ &= (A+B)\cos\alpha\cos\beta + (A-B)\sin\alpha\sin\beta \\ &= C\cos(\beta + \varepsilon). \end{aligned}$$

Homework : What is C and ε?

Resonance ($\omega = \omega_0$)

$mu'' + ku = F_0 \cos \omega_0 t$

$u = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \text{Re}(\frac{F_0}{2mi\omega_0} te^{i\omega_0 t}) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + c_3 t \sin \omega_0 t$

Try $u_p = p(t)e^{i\omega_0 t}$ s.t. $mu_p'' + ku_p = F_0 e^{i\omega_0 t}$

Take $\text{Re}(u_p)$

$mu_p'' = m(p'' + 2p'i\omega_0 - \omega_0^2 p)e^{i\omega_0 t} + kp\, e^{i\omega_0 t} = F_0 e^{i\omega_0 t}$

Hence $mp'' + 2mp'i\omega_0 = F_0$

and we get $p = \dfrac{F_0}{2mi\omega_0} t$

Take $\mathrm{Re}(u_p) = \dfrac{F_0}{2m\omega_0} t \cdot \sin(\omega_0 t)$,

$G(t,s) = \frac{\sin(\omega_0(t-s))}{\omega_0}$

$u_p(t) = \int_0^t \frac{1}{\omega_0} \sin(\omega_0(t-s)) \frac{F_0}{m} \cos(\omega_0 s) ds$

$mu'' + ru' + ku = F_0 \cos \omega_0 t$

$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + c_3 \cos(\omega_0 t + \delta)\,(\omega_0 \neq r_1\,, r_2)$

$\begin{cases} \mathrm{Re}\, r_1 < 0 \\ \mathrm{Re}\, r_2 < 0 \end{cases} \qquad \lim_{t\to\infty}(c_1 e^{r_1 t} + c_2 e^{r_2 t}) = 0$

HW : 3.7, 15, 17

HW : (1) $u'' - 3u' + 2u = \cos t$, find a particular solution.

(2) $u'' - 2u' + u = \cos t$, find a particular solution.