Subject : Differential Equation

Textbook : Boyce & Diprimai ; Elementary Differential Equation and Boundary Value Problems

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Chapter 1 Introduction

What is an Ordinary Differential Equation ?

What is an Partial Differential Equation ?

Find $u(t)$ such that $F(u, u', u'', \ldots, u^{(n)}, t) = 0$.

ODE : for example, $\dfrac{du}{dt} = u$.

NOT an ODE : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

More often ,we will consider ODE of the form

$$\frac{du}{dt} = f(t, u) \text{ with a given } f, \text{ or}$$
$$\frac{d^{(n)}u}{dt^{(n)}} = f\Big(t, u, \frac{du}{dt}, \ldots, \frac{d^{(n-1)}u}{dt^{(n-1)}}\Big).$$

We use the following notation :

$$\frac{du}{dt},\ u',\ \dot{u},\ u'(t),\ \dot{u}(t);$$
$$\frac{d^{(2)}u}{dt^{(2)}},\ u'',\ \ddot{u}(t);$$

sometimes we use $y(t),\ \dot{y},\ y'$; or $y(x),\ \dot{y},\ y'$.

Remark : ODE with parameters $u(t, m)$

$\frac{\partial u}{\partial t} = f(t, u, m)$,

$\frac{\partial u}{\partial t} = \frac{du}{dt}$, m is NOT involved in differentiation.

Order : the highest number of derivatives involved.

System of ODE : for example

$$\begin{cases} \frac{du}{dt} = f(u, v) \\ \frac{dv}{dt} = g(u, v) \end{cases}$$

Chapter 2 First Order Differential Equation

1st order linear ODE :

$$\frac{du}{dt} = f(t, u)$$

If $f(t, u) = p(t)u(t) + g(t)$ is linear in u,

then $\frac{du}{dt} = pu + g$ is a linear 1st order ODE.

I. Direct integration :

Eg: $\frac{dy}{dx} = \frac{-y}{2} + \frac{3}{2}$

$\Rightarrow \frac{dy}{dx} = \frac{3-y}{2}$, or $\frac{dy}{3-y} = \frac{dx}{2}$. $\cdots (*)$

Integrate both side, we get

$$\int \frac{dy}{3-y} = \int \frac{dx}{2}.$$

Hence $-\ln|3-y| = \frac{x}{2} + c$.

II. Integration factor :

$\frac{dy}{dx} + \frac{y}{2} = \frac{3}{2}$

$\mu(x)\frac{dy}{dx} + \mu(x)\frac{y}{2} = \frac{3}{2}\mu(x)$

[Chain Rule : $(fg)' = f'g + fg'$.]

Take $\mu(x) = \exp(\frac{x}{2})$, then $\frac{du}{dx} = \frac{1}{2}\exp(\frac{x}{2}) = \frac{1}{2}u(x)$.

It follows that

$$\exp\left(\frac{x}{2}\right)\frac{dy}{dx} + \left\{\frac{d}{dx}\exp\left(\frac{x}{2}\right)\right\}y = \frac{3}{2}\exp\left(\frac{x}{2}\right),$$

or

$$\left\{\exp\left(\frac{x}{2}\right)y\right\}' = \frac{3}{2}\exp\left(\frac{x}{2}\right).$$

Integrate both sides, we get $\exp(\frac{x}{2})\,y = 3\exp(\frac{x}{2}) + c$.

Therefore $y(x) = 3 + c\exp(\frac{-x}{2})$.

In general, a possible way to solve the first order linear equation,

$$y' + p(x)y = g(x)$$

is to multiply it by a suitable integrating factor $\mu(x)$. Then we have

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)g(x).$$

Now we want to choose $\mu(x)$ so that $\mu'(x) = p(x)\mu(x)$. Indeed,

$$\ln|\mu(x)| = \int p(x)dx + c$$

or

$$\mu(x) = c\exp\Big\{\int p(x)dx\Big\}.$$

Example:

$$\begin{cases} y' + ky = 0 \\ y(0) = y_0, \end{cases} \qquad \text{where } k, y_0 \text{ are some constants.} \tag{1}$$

- Method 1 : Direct integration

  Write the equation as $\dfrac{y'}{y} = -k$, then integrate both sides.

- Method 2 : Integrating factor

  Choose $\mu(x) = e^{kx}$, then $e^{kx}y' + ke^{kx}y = 0$, or $(e^{kx}y)' = 0$. Hence $y = ce^{-kx}$. And by $y(0) = y_0$, we have $c = y_0$.

- Remark: Suppose

  $$\begin{cases} y_1' + ky_1 = 0, \\ y_1(x_0) = y_1. \end{cases} \qquad \text{and} \qquad \begin{cases} y_2' + ky_2 = g, \\ y_2(x_0) = 0. \end{cases}$$

  Then $(y_1 + y_2)$ satisfies the following equation :

  $$\begin{cases} y' + ky = g, \\ y(x_0) = y_1. \end{cases}$$

- Eq.(1) has the solution $y(x) = y_0\, e^{-kx}$. Then $y(x) \to 0$ as $x \to +\infty$ in the case of $k > 0$. And $y(x) \to 0$ as $x \to -\infty$ in the case of $k < 0$.

Suppose $y' + ky = q$ for some constants k, q. Then

$$y' + k\Big(y - \frac{q}{k}\Big) = 0, \qquad \text{or} \quad \Big(y - \frac{q}{k}\Big)' + k\Big(y - \frac{q}{k}\Big) = 0.$$

Hence

$$\Big(y(x) - \frac{q}{k}\Big) = \Big(y(x_0) - \frac{q}{k}\Big)\, e^{-k(x-x_0)}.$$

If $k > 0$ ($k < 0$), as $x \to +\infty$ ($-\infty$), we have $y(x) \to \frac{q}{k}$. The solution of $k(y - \frac{q}{k}) = 0$ is called the equilibrium state. The equilibrium state of the equation $y' = f(y)$ is the set $\{y \,|\, f(y) = 0\}$.

Example:

$$\begin{cases} xy' + ky = 4x^2, \ k \text{ is a constant} \\ y(1) = 2 \end{cases}$$

Sol: We first find the integrating factor

$$\int \frac{k}{x} dx = k \ln|x| + c.$$

Multiply by $e^{k\ln|x|+c} = e^c(e^{\ln|x|})^k = |x|^k = x^k$, where we assume $x > 0$, then

$$x^k y' + kx^{k-1}y = 4x^{k+1}, \qquad \text{or } (x^k y)' = 4x^{k+1}.$$

Hence $x^k y = \dfrac{4\,x^{k+2}}{k+2} + c$; i.e. $y = \dfrac{4\,x^2}{k+2} + \dfrac{c}{x^k}$.

By the initial condition $y(1) = 2$, we get $c = \dfrac{2k}{k+2}$.

Generalizations :

Given $y' = f(y)$, we have

$$\frac{1}{f(y)}\frac{dy}{dx} = 1.$$

If we can find $G(y)$ such that $G'(x) = \frac{1}{f(x)}$, then

$$\frac{d}{dx}G(y(x)) = 1$$

and hence

$$G(y(x)) = x + c.$$

The solution is implicitly defined by G.

Generalize to separable case :

Consider the equation

$$M(x) + N(y)\frac{dy}{dx} = 0$$

If we can find $H_1'(x) = M(x)$ and $H_2'(y) = N(y)$, then

$$\frac{d}{dx}(H_1(x) + H_2(y(x)) = 0$$

The solution is implicitly defined by

$$H_1(x) + H_2(y(x)) = c$$

Example : Consider the differential equation

$$\begin{cases} y' = \dfrac{y\cos x}{1+2\,y^2} \\ y(0) = 1. \end{cases}$$

Sol:

$$\left(\frac{1+2y^2}{y}\right)y' = \cos x, \qquad \Longrightarrow \left(\frac{1}{y}+2y\right)y' = \cos x.$$

Then $(\ln|y| + y^2) = \sin x + c$, and $c = 1$ by the initial condition $y(0) = 1$.

Hence the solution $y(x)$ satisfies the relation $\ln|y| + y^2 = \sin x + 1$.

Integral curve

From the equation $\frac{dy}{dx} = F(x, y)$, given (x_0, y_0), we get the slope $F(x_0, y_0)$.

An integral curve is the curve $y = f(x)$ satisfies $\frac{dy}{dx} = F(x, f(x))$. But in some equation of the form $H_1(x) + H_2(y) = c$ (y is implicitly defined), the solution "$y = f(x)$" may not exists for all x.

$F(x_0, y_0) = 0 \Leftrightarrow$ tangent is horizontal.

$F(x_0, y_0) = \infty \Leftrightarrow$ tangent is vertical.

Equilibrium :

In the equation $y' = f(y)$, we say y_0 is an equilibrium iff $f(y_0) = 0$.

Existence and Uniqueness of solutions

Consider the equation

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)}, \\ y(0) = 1. \end{cases}$$

We have $2(y-1)dy = (3x^2 + 4x + 2)dx$. Integrate both sides, we obtain

$$y^2 - 2y = x^3 + 2x^2 + 2x + c, \qquad \text{and } c = -1.$$

Hence $y = 1 \pm \sqrt{x^3 + 2x^2 + 2x - 1}$, there are two solutions.

Example 2 :

Consider

$$\begin{cases} y' = y^{1/3} \\ y(x_0) = 0 \end{cases}.$$

Then $\frac{2}{3} y^{-1/3} y' = \frac{2}{3}$ or $(y^{2/3}) = \frac{2}{3}$.

Hence $y^{2/3} = \frac{2}{3}x + c$, and $c = -\frac{2}{3}x_0$ by the initial value $y(x_0) = 0$.

Consequently, $y = \pm[\, \frac{2}{3}(x - x_0)\,]^{2/3}$ for $x \geq x_0$.

If $x_0 = 0$, we get another solution $y \equiv 0$.

Example 3 :

Consider

$$\begin{cases} y' = y^2, \\ y(0) = y_0, \text{ where } y_0 \neq 0 \text{ is a constant.} \end{cases}$$

Then $-y^{-2} y' = -1$, or $(\frac{1}{y})' = -1$. Therefore

$$\frac{1}{y} = -x + \frac{1}{y_0}, \qquad \text{or} \qquad y(x) = \frac{y_0}{1 - y_0\, x}.$$

- If $y_0 > 0$, then $y(x) \to +\infty$ as $x \to (\frac{1}{y_0})^-$, and the solution exists only on $(-\infty, \frac{1}{y_0})$.

- If $y_0 = 0$, then $y \equiv 0$ is the solution defined on the whole real line.

- If $y_0 < 0$, then $y(x) \to -\infty$ as $x \to (\frac{1}{y_0})^+$, and the solution exists only on $(\frac{1}{y_0}, \infty)$.

These nonuniqueness or non-existence will not happen to linear first order ODE's.

Theorem: If $p(t)$, $g(t)$ are continuous on $\alpha < t < \beta$ and $t_0 \in (\alpha, \beta)$, then there exists the unique solution on (α, β) for the equation

$$\begin{cases} y' + p(t)y = g(t), \\ y(t_0) = y_0. \end{cases}.$$

Moreover, the solution is given by

$$y(t) = \frac{1}{\mu(t)}\left[\mu(t_0)y(t_0) + \int_{t_0}^{t} \mu(s)g(s)\,\mathrm{d}s\right],$$

where $\mu(t) = \exp\{\int_{t_0}^{t} p(s)\,\mathrm{d}s\}$.

Proof:

Let $\mu(t) = \exp\{\int_{t_0}^{t} p(s)\,\mathrm{d}s\}$ on (α, β).

Then $\mu'(t) = p(t)\mu(t)$.

Multiply $\mu(t)$ on the both side of the equation, we get

$$\mu(t)y'(t) + p(t)\mu(t)y(t) = \mu(t)g(t).$$

That is

$$(\mu(t)y(t))' = \mu(t)g(t).$$

Hence

$$(\mu y)(t) = (\mu y)(t_0) + \int_{t_0}^{t} \mu(s) g(s)\, \mathrm{d}s,$$

and $y(t) = \frac{1}{\mu(t)}[\,\mu(t_0)y(t_0)\,] + \int_{t_0}^{t} \mu(s)g(s)\,\mathrm{d}s.$

Theorem : For nonlinear case

$$\begin{cases} y' = F(t, y) \\ y(t_0) = y_0. \end{cases}$$

If F and $\frac{\partial F}{\partial y}$ are continuous in a neighborhood of (t_0, y_0) then there exists unique solution on the neighborhood.

Asymptotic Stability of Equilibrium States

Recall : If y_0 is the only solution to $f(y) = 0$, and $y' = f(y)$, then

$$\lim_{t\to\infty} y(t) = y_0, \qquad \text{and} \qquad \begin{cases} f(y) < 0, \text{ if } y > y_0; \\ f(y) > 0, \text{ if } y < y_0. \end{cases}$$

Growth rate proportional to population

Modeling with $y' = h(y)\, y$:

$$\text{Require} \begin{cases} h(y) \sim r, \text{ if } y \text{ is small;} \\ h(y) \text{ decrease as } y \text{ grows ;} \\ h(y) < 0, \text{ when } y > k, \text{ for some constant} k. \end{cases}$$

For example,

$$h(y) = r - \frac{r}{k}\, y = r\Big(1 - \frac{1}{k} y\Big).$$

$$\text{i.e.} \begin{cases} y' = r(1 - \dfrac{y}{k})y; \\ y(0) = c. \end{cases}$$

Remark : this ODE is solvable by directly integrate $\dfrac{\mathrm{d}y}{r\left(1-\dfrac{y}{k}\right)y} = \mathrm{d}t.$

Logistic Growth

Qualitative analysis of $y' = r\left(1-\dfrac{y}{k}\right)y$, (without solving it explicitly)

1. Equilibrium states are $y = 0$, and $y = k$.

2. if $0 < y < k$, then $r(1-\frac{y}{k})y > 0$.

3. if $y > k$, then $r(1-\frac{y}{k})y < 0$.

Thus $y = k$ is an asymptotically stable equilibrium.

That is $\lim\limits_{t\to\infty} y = k$, if $|\,y(0) - k\,|$ is small.

But "0" is an unstable equilibrium state.

That is $\lim\limits_{t\to\infty} y = 0$ only if $y(0) = 0$.

Threshold

Consider the equation

$$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)y,$$

where r, T are given positive constants.

1. $y = 0$ and $y = T$ are the critical points, corresponding to the equilibrium solutions $y_1(t) = 0$ and $y_2(t) = T$.

2. If $0 < y < T$, then $\frac{dy}{dt} < 0$, and $y(t)$ decreases as t increases. Thus $y_1(t) = 0$ is an asymptotically stable equilibrium solution.

3. If $y > T$, then $\frac{dy}{dt} > 0$. Thus $y_2(t) = T$ is an unstable equilibrium solution.

From this, T is called threshold.

Threshold + Logistic

Consider $f(y) = -(1 - \frac{y}{T})(1 - \frac{y}{K})y$, $(\, 0 < T < K \,)$ then

1. $y(0) < T \Longrightarrow \lim_{t\to\infty} y(t) = 0$,

2. $y(0) > T \Longrightarrow \lim_{t\to\infty} y(t) = K$.

Exact Equations

Let the differential equation

$$M(x, y) + N(x, y)y' = 0 \tag{2}$$

be given. Suppose we can identify a function ψ such that

$$\psi_x(x, y) = M(x, y),\ \psi_y(x, y) = N(x, y),$$

and such that $\psi(x, y) = c$ defines $y = \phi(x)$ implicitly as a differentiable function of x.

Then

$$M(x,y) + N(x,y)y' = \psi_x(x,y) + \psi_y(x,y)y' = \frac{d}{dx}\psi(x,\phi(x))$$

and the equation becomes

$$\frac{d}{dx}\psi(x,\phi(x)) = 0.$$

In this case Eq.(2) is said to be an **exact** differential equation.

The solution is given implicitly by

$$\psi(x,y) = c,$$

for some constant c.

Theorem: Let $M, N \in C^1(\Omega)$, $\Omega \in \mathbb{R}^2$ is simply connected. Then there exists a function $\psi(x,y)$ such that

$$\psi_x(x,y) = M(x,y);\ \psi_y(x,y) = N(x,y),$$

if and only if $M_y(x,y) = N_x(x,y)$.

Integrating Factors

Let us multiply the equation

$$M(x,y)dx + N(x,y)dy = 0$$

by a function μ and then try to choose μ so that

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0 \tag{3}$$

is exact. By the above Theorem, we need

$$(\mu M)_y = (\mu N)_x.$$

Hence, the integrating factor μ must satisfy the equation

$$M\mu_y - N\mu_x + \mu(M_y - N_x) = 0.$$

If such μ can be found, then Eq.(3) will be exact.

Remark: The solution μ may have more than one solution. But, unfortunately, μ is difficult to solve. The situations in which integrating factors can be found occur when μ is a function of only one of the variables x, y.

Assume that $\mu = \mu(x)$, we have

$$(\mu M)_y = \mu M_y, \qquad (\mu N)_x = \mu N_x + N\frac{d\mu}{dx}.$$

Thus, $(\mu M)_y = (\mu N)_x$ is equivalent to

$$\frac{d\mu}{dx} = \Big(\frac{M_y - N_x}{N}\Big)\mu. \tag{4}$$

If $\dfrac{M_y - N_x}{M}$ is a function of x only, then $\mu(x)$ can be found by solving the 1st order linear equation (4).

Homework : Try $\mu = \mu(y)$.

Example : $(3xy + y^2) + (x^2 + xy)y' = 0$

Observe that

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{1}{x}.$$

Thus, μ is a function of x only, and

$$\frac{d\mu}{dx} = \frac{\mu}{x}.$$

Hence

$$\mu(x) = x.$$

Example : $(3x^2y + xy^2) + (x^2y + x^3)y' = 0$

Note that $M(x,y) = 3x^2y + xy^2$, $N(x,y) = x^2y + x^3$, and

$$M_y(x,y) = 3x^2 + 2xy = N_x(x,y).$$

Want to find $\psi(x,y)$ s.t. $\psi_y = x^2y + x^3$ and $\psi_y = 3x^2y + xy^2$.

Hence $\psi(x,y) = \frac{x^2y^2}{2} + x^2y + F(x)$ and $\psi(x,y) = x^3y + \frac{x^2y^2}{2} + G(y)$.

Therefore $\psi(x,y) = \frac{x^2y^2}{2} + x^3 + \text{constant}$.

Answer : $\dfrac{x^2y(x)^2}{2} + x^3y(x) = \text{constant}$.

Existence and Uniqueness

We first prove the **uniqueness** of the solution as follow :

Suppose the initial value problem

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases} \tag{5}$$

has two solutions y_1 and y_2.

Let $w(t) = y_1(t) - y_2(t)$, then

$$\begin{cases} y_1'(t) = f\big[t, y_1(t)\big], \quad y_1(t_0) = y_0; \\ y_2'(t) = f\big[t, y_2(t)\big], \quad y_2(t_0) = y_0. \end{cases}$$

and $w'(t) = f\big[t, y_1(t)\big] - f\big[t, y_2(t)\big]$, and $w(t_0) = 0$.

Hence

$$w(t) = \int_{t_0}^{t} \{f[s, y_1(s)] - f[s, y_2(s)]\}\, \mathrm{d}s,$$

and

$$|\, w(t)\, | \le \int_{t_0}^{t} |\, f(s, y_1) - f(s, y_2)\, |\, \mathrm{d}s.$$

If df/dy is continuous,

$$\left|\, \frac{df}{dy}\, \right| \le M \qquad \text{on the region } \Omega = \Big\{(t, y)\, \Big|\, |\, t - t_0\, | \le a, |\, y - y_0\, | \le b\Big\},$$

and we have

$$\Big|\, f(s, y_1) - f(s, y_2)\, \Big| = \Big|\, \int_{y_1}^{y_2} \frac{\partial f}{\partial y}(s, y) dy\, \Big| \le M|\, y_1 - y_2\, |.$$

It follows that

$$|\, w(t)\, | \le M \int_{t_0}^{t} |\, w(s)\, | ds.$$

Define $U(t) = \int_{t_0}^{t} |\, w(s) \,| ds$, then

$$U(t) \geq 0, \text{ for } t \geq t_0 \tag{6}$$

On the other hand,

$$U'(t) \leq |\, w(t) \,|,$$

and since $|\, w(t) \,| \leq M \int_{t_0}^{t} |\, w(s) \,| ds = MU(t)$, we have

$$U' \leq MU, \qquad \text{or } U' - MU \leq 0.$$

Therefore,

$$(e^{-Mt}U)' \leq 0 \qquad \text{for } t \geq t_0.$$

Since $U(t_0) = 0$, we obtain

$$U(t) \leq 0, \qquad \text{for } t \geq t_0. \tag{7}$$

by (6) and (7), $U(t) \equiv 0$; and hence

$$w(t) \equiv 0, \qquad \text{for } t \geq t_0 \text{ on } \Omega.$$

Homework: Show the same result for $t \leq t_0$ on Ω.

Theorem: If f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle Ω, then there is some interval $|\, t - t_0 \,| \leq h < a$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem(5).

Existence of solution:

The method we use to find the solution is known as **Picard's iteration method**. First, let $y_0(t) = y_0$; and

$$y_1(t) = y_0 + \int_{t_0}^{t} f(s, y_0(s))ds.$$

Similarly, y_2 is obtained from y_1:

$$y_2(t) = y_0 + \int_{t_0}^{t} f(s, y_1(s))ds,$$

and in general,

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))ds.$$

In this manner we generate the sequence of functions $\{y_n \Big| n = 0, 1, 2, \cdots\}$.

1. All y_n's exist on the rectangle $D = \Big\{(t, y) \Big| |t - t_0| \leq h, |y - y_0| \leq b\Big\}$, where $h = \min\{a, \frac{b}{m}\}$ and $M = \max_{(t,y)\in\Omega} |f(t, y)|$.

2. $y_n(t)$ converges.

   We first estimate $\Big| y_{n+1}(t) - y_n(t) \Big|$ as follows:

$$\begin{aligned} \Big| y_{n+1}(t) - y_n(t) \Big| &\leq \int_{t_0}^{t} \Big| f(s, y_n(s)) - f(s, y_{n-1}(s)) \Big| ds \\ &\leq \int_{t_0}^{t} \int_{y_{n-1}(s)}^{y_n(s)} \Big| \frac{\partial f}{\partial y}(s, \xi) \Big| \xi ds \\ &\leq K \int_{t_0}^{t} \Big| y_n(s) - y_{n-1}(s) \Big| ds \end{aligned}$$

where $K = \max_{(t,y)\in\Omega} \left| \frac{\partial f}{\partial y}(t, y) \right|$. And hence

$$\begin{aligned}
|\, y_1(t) - y_0(t) \,| &\leq M \,|\, t - t_0 \,|, \\
|\, y_2(t) - y_1(t) \,| &\leq K \textstyle\int_{t_0}^t M(s - t_0) ds = KM \frac{|\, t - t_0 \,|^2}{2}, \\
|\, y_3(t) - y_2(t) \,| &\leq K \textstyle\int_{t_0}^t KM(s - t_0)^2 ds = K^2 M \frac{|\, t - t_0 \,|^3}{3!}, \\
&\vdots \\
|\, y_{n+1}(t) - y_n(t) \,| &\leq K^n M \frac{|\, t - t_0 \,|^{n+1}}{(n+1)!} \leq K^n M \frac{h^{n+1}}{(n+1)!}.
\end{aligned}$$

From this, it is easy to show that the sequence $\{y_n\}$ converges uniformly on $(t_0 - h, t_0 + h)$, and we denote the limit function by y^*.

Homework: Show that $\int_{t_0}^t f(s, y_n(s)) ds \to \int_{t_0}^t f(s, y^*(s))\, \mathrm{d}s$ as $n \to \infty$.

3. Since $y_{n+1}(t) = y_0 + \int_{t_0}^t f(s, y_n(s))\, \mathrm{d}s$, and $\{y_n\}$ converges to $y*$, we obtain

$$y^*(t) = y_0 + \int_{t_0}^t f(s, y^*(s))\, \mathrm{d}s \qquad \text{on } (t_0 - h, t_0 + h).$$

Consequently, y^* is a solution to (5).