Solutions to Problem Set 1

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Problem 2.4 (a)

$$[\mathbf{A}|b]_{\mathrm{REFF}} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -2 \end{array}\right)$$

Thus, because $\mathrm{rank}(\mathbf{A}) = \mathrm{rank}(\mathbf{A}|b) = 3$, the solution is $(x, y, z) = (0, \frac{3}{2}, -1)$, and this solution is unique.

(b)

$$[\mathbf{A}|b]_{\mathrm{REFF}} = \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{3}{2} & -\frac{2}{3} \\ 0 & 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & -6 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

Here $\mathrm{rank}(\mathbf{A}) = \mathrm{rank}(\mathbf{A}|b) = 4 < 6$, and the solution set can be expressed in terms of 2 free parameters.

Problem 2.5 (a)

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{pmatrix} \qquad \mathbf{U} = \begin{pmatrix} 3 & 2 \\ 0 & -\frac{1}{3} \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

(b)

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & \frac{3}{4} & 1 \end{pmatrix} \qquad \mathbf{U} = \begin{pmatrix} 2 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1\frac{3}{4} \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{7}\begin{pmatrix} -3 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & -3 \end{pmatrix}$$

Problem 2.7

$$\mathbf{expm}(\mathbf{A}) = \begin{pmatrix} 54.3971 & -47.0080 & -47.0080 \\ -47.0080 & 54.3971 & 47.0080 \\ -47.0080 & 47.0080 & 54.3971 \end{pmatrix}$$

$$\mathbf{expm}(-\mathbf{A}) = \begin{pmatrix} 0.0925 & 0.0429 & 0.0429 \\ 0.0429 & 0.0925 & -0.0429 \\ 0.0429 & -0.0429 & 0.0925 \end{pmatrix}$$

and

$$\mathbf{expm}(\mathbf{A}) * \mathbf{expm}(-\mathbf{A}) = \begin{pmatrix} 1.0000 & 0.0000 & 0.0000 \\ -0.0000 & 1.0000 & -0.0000 \\ -0.0000 & 0.0000 & 1.0000 \end{pmatrix}$$

Problem 3.1 (a) (i) Since $\mathbf{R}$ is an orthogonal matrix, we have $|\hat{e}_k| = |\mathbf{R}\hat{e}_k| = |\lambda_k \hat{e}_k| = |\lambda_k||\hat{e}_k|$. As an eigenvector, $|\hat{e}_k| \neq 0$, thus $|\lambda_k| = 1$, i.e., $\lambda_k = \pm 1$.

(ii) $\mathbf{R}\mathbf{R}^T = \mathbf{I} \Rightarrow \det(\mathbf{R})\det(\mathbf{R}^T) = \det(\mathbf{R})^2 = 1 \Rightarrow \det(\mathbf{R}) = \pm 1$

(b) The proof is straightforward.

(c) (i) $\det(\mathbf{R}) = \cos^2\theta + \sin^2\theta = 1$.

(ii) $\mathbf{P}$ reflects a vector with respect to the xy plane.

(iii) Consider an arbitrary rotation $\mathbf{R}(\delta\theta)$ by an *infinitesimal* angle $\delta\theta$, then since each entry of $\mathbf{R}(\theta)$ is a continuous function of θ, we have $\mathbf{R}(\delta\theta) - \mathbf{I} = \mathbf{R}(\delta\theta) - \mathbf{R}(0) = \mathcal{O}(\delta\theta)$, where $\mathbf{I}$ is the identity matrix, and hence $\det \mathbf{R}(\delta\theta) = 1$. Now since $\mathbf{R}(\theta) = \mathbf{R}(n\frac{\theta}{n}) = \mathbf{R}^n(\frac{\theta}{n})$, where n can be made arbitrarily large, we have $\det \mathbf{R}(\theta) = (\det \mathbf{R}(\frac{\theta}{n}))^n = 1$

(iv) Suppose that $\mathbf{A}\hat{e} = \lambda\hat{e}$, let $\hat{e}' = \mathbf{R}\hat{e}$, then $\mathbf{A}'\hat{e}' = \mathbf{R}\mathbf{A}(\mathbf{R}^T\mathbf{R})\hat{e} = \mathbf{R}(\mathbf{A}\hat{e}) = \lambda(\mathbf{R}\hat{e}) = \lambda\hat{e}'$, i.e., λ is also an eigenvalue of $\mathbf{A}'$.

Problem 3.3 (a)

$$\hat{e}_0 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

(b)

$$\mathbf{P}_0 = \hat{e}_0\hat{e}_0^T = \frac{1}{6} \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix}$$

and direct computation shows that $\mathbf{P}_0\mathbf{A} = 0$. This is always true under the assumption that $\mathbf{A}$ is a real symmetric matrix. The proof is as follows: since $\mathbf{A}$ is real–symmetric, $\mathbf{A}$ has a spectral decomposition, i.e., $\mathbf{A} = \sum_{k=0}^{n-1} \lambda_k \mathbf{P}_k$. Using the property of a projection operator, namely $\mathbf{P}_k\mathbf{P}_{k'} = \mathbf{P}_k\delta_{kk'}$, we have $\mathbf{P}_0\mathbf{A} = \lambda_0\mathbf{P}_0 = 0$. A counterexample like $\mathbf{A} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ shows that generally this's not true if $\mathbf{A}$ is not real–symmetric.

Problem 2.11 (a)

$$\mathbf{A}\vec{x}_1 = \left(\begin{array}{c} 10 \\ 5+\epsilon \end{array}\right) \Rightarrow \|\mathbf{A}\|_2 \simeq \frac{|\mathbf{A}\vec{x}_1|}{|\vec{x}_1|} = \sqrt{25 + 2\epsilon + \frac{\epsilon^2}{5}}$$

$$\mathbf{A}^{-1}\vec{x}_2 = \frac{1}{4\epsilon}\left(\begin{array}{c} 5+\epsilon \\ -10 \end{array}\right) \Rightarrow \|\mathbf{A}^{-1}\|_2 \simeq \frac{|\mathbf{A}^{-1}\vec{x}_2|}{|\vec{x}_2|} = \frac{1}{4\epsilon}\sqrt{25 + 2\epsilon + \frac{\epsilon^2}{5}}$$

(b)

$$K = \|\mathbf{A}^{-1}\|\|\mathbf{A}\| \simeq \frac{1}{4\epsilon}(25 + 2\epsilon + \frac{\epsilon^2}{5})$$

(c)

$$\frac{1}{K}\frac{|\delta\vec{b}|}{|\vec{b}|} \leq \frac{|\delta\vec{x}|}{|\vec{x}|} \leq K\frac{|\delta\vec{b}|}{|\vec{b}|}$$

Let

$$\frac{|\delta\vec{b}|}{|\vec{b}|} = \varepsilon_{\text{mach}}$$

then

$$\frac{1}{K}\varepsilon_{\text{mach}} \leq \frac{|\delta\vec{x}|}{|\vec{x}|} \leq K\varepsilon_{\text{mach}}$$

Extra credit: It's easy to prove the following results:

Let $\mathbf{A} = (a_{ij})_{m\times n}$ be any $m \times n$ matrix, then $\|\mathbf{A}\|_1$ is equal to the "maximum column sum" of $\mathbf{A}$, i.e.,

$$\|\mathbf{A}\|_1 = \max_{1\leq j\leq n}\sum_{i=1}^{m}|a_{ij}|$$

and $\|\mathbf{A}\|_\infty$ is equal to the "maximum row sum" of $\mathbf{A}$, i.e.,

$$\|\mathbf{A}\|_\infty = \max_{1\leq i\leq m}\sum_{j=1}^{n}|a_{ij}|$$