Solutions to Problem Set 1

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Problem 1.1(b)

$$\begin{aligned} 2x_1 - x_2 \quad &= 1 \\ -x_1 + 2x_2 - x_3 &= 1 \\ -x_2 + 2x_3 &= 1 \end{aligned}$$

Step 1: First–column elimination.

$$\begin{aligned} 2x_1 - x_2 \quad &= 1 \\ +3x_2 - 2x_3 &= 3 \\ -x_2 + 2x_3 &= 1 \end{aligned}$$

Step 2: Second–column elimination.

$$\begin{aligned} 2x_1 - x_2 \quad &= 1 \\ +3x_2 - 2x_3 &= 3 \\ +2x_3 &= 3 \end{aligned}$$

Step 3: Backward substitution.

$$\begin{aligned} x_1 &= \frac{3}{2} \\ x_2 &= 2 \\ x_3 &= \frac{3}{2} \end{aligned}$$

Problem 1.3

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 + x_5 &= 4 \\ 2x_1 + x_2 + 2x_3 + x_4 \quad &= 1 \\ x_1 + 2x_2 + x_3 \quad + 2x_5 &= 0 \end{aligned}$$

Step 1: First–column elimination.

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 + x_5 &= 4 \\ -3x_2 \quad - x_4 - 2x_5 &= -7 \\ -x_4 + x_5 &= -4 \end{aligned}$$

Step 2: Switch columns 3 and 4.

$$\begin{aligned} x_1+2x_2+x_4+x_3+\ x_5 &= 4 \\ -3x_2-x_4 \qquad -2x_5 &= -7 \\ -x_4 \qquad +\ x_5 &= -4 \end{aligned}$$

Step 3: Move terms involving x_3 and x_5 to the right side.

$$\begin{aligned} x_1+2x_2+x_4 &= +4-x_3-x_5 \\ -3x_2-x_4 &= -7 \qquad +2x_5 \\ -x_4 &= -4 \qquad -x_5 \end{aligned}$$

Step 4: Backward substitution.

$$\begin{aligned} x_1 &= -2 - x_3 \\ x_2 &= 1 - x_5 \\ x_4 &= 4 + x_5 \end{aligned}$$

Step 5: Clearly, if we set x_3 to be $-\alpha$ and x_5 to be $-\beta$, we would obtain the desired form for the solution, i.e.:

$$\begin{aligned} x_1 &= \alpha - 2 \\ x_2 &= \beta + 1 \\ x_3 &= -\alpha \\ x_4 &= 4 - \beta \\ x_5 &= -\beta \end{aligned}$$

Problem 1.4

$$\begin{aligned} 2x_1+\ x_2+2x_3 &= c \\ x_1+2x_2+\ x_3 &= 1 \\ 2x_1+\ x_2+2x_3 &= 0 \end{aligned}$$

(a) **Step 1:** First column elimination.

$$\begin{aligned} 2x_1+\ x_2+2x_3 &= c \\ 3x_2 \qquad &= 2 - c \\ 0x_3 &= c \end{aligned}$$

The third equation shows that a solution is possible *only* if $c = 0$.

(b) With $c = 0$, set x_3 to be $-\gamma$ and solve the above equations by backward substitution, there results:

$$\begin{aligned} x_1 &= \gamma - \frac{1}{3} \\ x_2 &= \frac{2}{3} \\ x_3 &= -\gamma \end{aligned}$$

Problem 1.7 (a) The following four matrices form a basis of the space of 2×2 real matrices:

$$\sigma_1 = \mathbf{A}, \qquad \sigma_2^{'} = i\sigma_2, \qquad \sigma_3, \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where σ_1, σ_2 and σ_3,are *Pauli matrices* defined in part b. Suppose that $\mathbf{B} = a_1\sigma_1 + a_2\sigma_2^{'} + a_3\sigma_3 + a_4\mathbf{I}$ where a_i, $i = 1, \ldots, 4$ are unknown real coefficients, then a simple computation shows that $\mathbf{B}$ anticommutes with $\mathbf{A}$ if and only if $a_1 = 0$ and $a_4 = 0$. The general solution is thus of the form $a\sigma_2^{'} + b\sigma_3$, $a, b \in \mathbb{R}$, $a \neq 0$.

(b) Straightforward computation will do the job.

Problem 1.14 (a) Clearly, the following relation holds for two basis:

$$\begin{pmatrix} \hat{i}^{'} \\ \hat{j}^{'} \\ \hat{k}^{'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}$$
$$:= \mathbf{U} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}$$

where the second equation defines the matrix $\mathbf{U}$. $\mathbf{U}$ is an orthogonal matrix, i.e., $\mathbf{U}^T\mathbf{U} = 1$. Furthermore, we have

$$\begin{aligned} \vec{a} &= \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} \\ &= \begin{pmatrix} a_1^{'} & a_2^{'} & a_3^{'} \end{pmatrix} \begin{pmatrix} \hat{i}^{'} \\ \hat{j}^{'} \\ \hat{k}^{'} \end{pmatrix} \\ &= \begin{pmatrix} a_1^{'} & a_2^{'} & a_3^{'} \end{pmatrix} \mathbf{U} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} \end{aligned}$$

Hence,

$$\begin{pmatrix} a_1^{'} & a_2^{'} & a_3^{'} \end{pmatrix} \mathbf{U} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}$$

then

$$\mathbf{U}^T \begin{pmatrix} a_1^{'} \\ a_2^{'} \\ a_3^{'} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and

$$\begin{pmatrix} a_1^{'} \\ a_2^{'} \\ a_3^{'} \end{pmatrix} = (\mathbf{U}^T)^{-1} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$
$$= \mathbf{U} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

(b) From now on, we take an active viewpoint of the rotation, namely, the coordinate system is fixed but the vector is rotating. It's easy to check that the new vector $\vec{a}^{'}$ is related to the original one $\vec{a}$ by the same matrix $\mathbf{U}(\theta)$. Clearly, the matrix $\mathbf{V}$ which will rotate the vector back to its original position is $\mathbf{U}(\theta)^{-1}$, i.e.,

$$\begin{aligned} \mathbf{V} &= \mathbf{U}(\theta)^{-1} \\ &= \mathbf{U}(\theta)^{T} \\ &= \mathbf{U}(-\theta) \\ &= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

(c) Sample MATLAB code:

```
n=4;
while n <= 16
   theta = 2*pi/n;
   n
   U=[cos(theta),sin(theta),0;-sin(theta),cos(theta),0;0,0,1]
   U^n
   n = 2*n;
end
```

Problem1.16 (a)

$$\hat{e}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\hat{e}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$$

$$\hat{e}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$$

$$\hat{e}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$$

(b) The generalization is immediate: if the given n vectors are

$$x_i(k) = \begin{cases} 1, & 1 \le k \le i \\ 0, & i < k \le n. \end{cases}$$

then the resulted orthogonal unit vectors by the Gram–Schmidt orthogonalization method are $\hat{e}_i(k) = \delta_{ik}$, $i = 1, \ldots, n$. The proof is by induction.

(c) No, it does not.