Brief Solution to Midterm 02

Dec 09, 2020.

1. (10 pts) State (need not prove) the error formula of Lagrangian interpolation for smooth functions defined on $[a,b]$ with data given on uniformly spaced nodes $x_0, \cdots, x_n$ on $[a,b]$ where $x_j = a + jh$, $h = (b-a)/n$.

   Ans:

   Let $P_n(x)$ be the nth Lagrange interpolating polynomail. The error formula is given by
   $$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$
   where $\xi(x) \in (a,b)$.

2. (12 pts) Let P_n be the degree n interpolating polynomial of $f(x) = \dfrac{1}{1+x^2}$ on the uniformly spaced nodes $x_0, \cdots, x_n$ on $[-5,5]$ with $x_j = -5 + jh$, $h = 10/n$. Is it true that
   $$\max_{-5 \le x \le 5} |f(x) - P_n(x)| \to 0 \quad \text{as } n \to \infty?$$
   Use either analytic proof or numerical evidence to support your claim.

   Ans: No. $f^{(n)}$ grows fast in n. The error formula does not guarantee convergence **(4 pts)**.

   The interpolating polynomial oscillates wildly on $[-5,5]$ for $x \neq x_i$ as n increases. Some demonstration of the graph or interpolating polynomial value at a fixed $x \neq x_i$ with $n \sim 10-20$ will do **(8 pts)**.

3. (12 pts) Suppose that we are to construct a piecewise polynomial interpolation $S(x)$ such that $S(x_i) = f(x_i)$, $S'(x_i) = f'(x_i)$, $i = 0, \cdots, n$ with additional continuity conditions for S'' and S''' on the interior nodes $x_1, \cdots, x_{n-1}$. If we use polynomials of the same degree on each of the interval $[x_0, x_1], \cdots, [x_{n-1}, x_n]$, what is the minimal degree needed in each interval? How many additional end conditions are needed? Count carefully and explain (give details).

   Ans: Values of S at x_0, x_n: one condition each.

   Values of S at $x_1, \cdots x_{n-1}$: two conditions each.

   Values of S' at x_0, x_n: one condition each.

   Values of S' at $x_1, \cdots x_{n-1}$: two conditions each.

   Continuity of S'' at $x_1, \cdots x_{n-1}$: one condition each.

   Continuity of S''' at $x_1, \cdots x_{n-1}$: one condition each.

   (6 pts)

Total $6n-2$ conditions. Therefore we require minimal $6n$ unknowns or degree 5 polynomials on each interval and additional 2 boundary conditions **(6 pts)**.

4. (12 pts) Let $x_i = 0.01*i$. Use all or part of the data $(x_0, \sin(x_0)), (x_1, \sin(x_1)), \cdots, (x_{100}, \sin(x_{100}))$ and inverse cubic spline interpolation method to find an approximate value of $\sin^{-1}(0.3)$. Partial credit if you use a different interpolation method.

Ans:

$x = 0 : .01 : 1$; $y = \sin(x)$.

Answer $= \text{spline}(y, x, 0.3) = 0.3046926539299876$.

5. (12 pts) Suppose f is smooth and the data $f(x)$, $f(x\pm h)$, $f(x\pm 2h), \cdots$ are prescribed.

   (a) Find a second order approximation of $f'(x+\frac{h}{2})$ with minimal number of data points and derive an error identity of the form $f'(x+\frac{h}{2})-f'_h(x+\frac{h}{2}) = C_1 f^{(n_1)}(\xi_1)h^2$.

   (b) Find a fourth order approximation of $f'(x+\frac{h}{2})$ with minimal number of data points and derive an error bound of the form $|f'(x+\frac{h}{2}) - f'_h(x+\frac{h}{2})| \le C_2|f^{(n_2)}(\xi_2)|h^4$.

Ans:

(a) Apply Taylor expansion to $f(x+h)$ and $f(x)$ around $f(x+\frac{h}{2})$ leads to

$$\frac{f(x+h)-f(x)}{h} = f'(x+\frac{h}{2}) + \frac{h^2}{24}f'''(\xi))(\mathbf{4pts}).$$

(b) Also

$$\frac{f(x+h)-f(x)}{h} = f'(x+\frac{h}{2}) + \frac{h^2}{24}f'''(x+\frac{h}{2}) + \frac{h^4}{1920}f^{(5)}(\xi_1)$$

It follows that $\frac{f(x+2h)-f(x-h)}{3h}$ admits a similar estimate with $h = (x+h)-x$ replaced by $3h = (x+2h)-(x-h)$:

$$\frac{f(x+2h)-f(x-h)}{3h} = f'(x+\frac{h}{2}) + \frac{(3h)^2}{24}f'''(x+\frac{h}{2}) + \frac{(3h)^4}{1920}f^{(5)}(\xi_2)$$

Therefore

$$f'_h(x+\frac{h}{2}) = \frac{9}{8}\frac{f(x+h)-f(x)}{h} - \frac{1}{8}\frac{f(x+2h)-f(x-h)}{3h}(\mathbf{4pts}).$$

with

$$f'_h(x+\frac{h}{2}) = f'(x+\frac{h}{2}) + E_h$$

and

$$|E_h| \le \frac{3}{512}h^4|f^{(5)}(\xi)|(\mathbf{4pts}).$$

6. (12 pts) Give formula for composite trapezoidal rule, midpoint rule and Simpson rule using $(x_i, f(x_i))$, $i = 0\cdots, n$, where $x_j = a + jh$, $h = (b-a)/n$ (you can use $(x_{i-1/2}, f(x_{i-1/2}))$, $i = 1, \cdots, n$ for midpoint rule if you prefer). Give corresponding error formula of the form: $I_h(a,b) - I(a,b) = C\sum_i h^p f^{(q)}(\xi_i)$ (find C and p, q for composite trapezoidal and midpoint rule, need not derive) and $|I_h(a,b) - I(a,b)| \le \sum_i Ch^p|f^{(q)}(\xi_i)|$. (find only p, q for composite Simpson's rule). Need not derive if you are sure about the answer.

Ans:

Composite Trapezoidal Rule:

$$\int_a^b f(x)dx = \frac{h}{2}\Big[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\Big] - \frac{b-a}{12}f''(\mu)h^2$$

where n is an integer, $h = \frac{b-a}{n}$, $x_j = a + jh$, for $j = 0, 1, ..., n$, and some $\mu \in (a,b)$ **(4 pts)**

Composite Midpoint Rule:

$$\int_a^b f(x)dx = 2h\sum_{j=1}^{n/2} f(x_{2j-1}) + \frac{b-a}{6}f''(\mu)h^2$$

where n is an integer, $h = \frac{b-a}{n}$, $x_j = a + jh$, for $j = 0, 1, ..., n$, and some $\mu \in (a,b)$ **(4 pts)**

Composite Simpson's Rule:

$$\int_a^b f(x)dx = \frac{h}{3}\Big[f(a) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + f(b)\Big] - \frac{b-a}{180}f^{(4)}(\mu)h^4$$

where n is an integer, $h = \frac{b-a}{n}$, $x_j = a + jh$, for $j = 0, 1, ..., n$, and some $\mu \in (a,b)$ **(4 pts)**

7. (15 pts) Use composite trapezoidal rule $N_1(h)$to evaluate $\int_0^1 x^{0.5}dx$ and find the rate of convergence (p in Ch^p) numerically. Then use Richardson extrapolation to derive $N_2(h)$ and find the corresponding p for $N_2(h)$ numerically.

Ans:

$$I = N_1(h) + Ch^p + \cdots$$

$$I = N_1(2h) + C(2h)^p + \cdots$$

$$p \approx \log_2 \frac{I - N_1(2h)}{I - N_1(h)}$$

Numerical result shows that $p = 1.5$. **(3 pts)**

Therefore

$$I = N_1(2h) + C(2h)^{\frac{3}{2}} + \cdots$$

or

$$I = N_1(\frac{h}{2}) + C(\frac{h}{2})^{\frac{3}{2}} + \cdots$$

Use either $N_1(h)$ and $N_1(2h)$ or $N_1(h)$ and $N_1(\frac{h}{2})$ to get

$$N_2(h) = \frac{2\sqrt{2}}{2\sqrt{2}-1} N_1(h) - \frac{1}{2\sqrt{2}-1} N_1(2h)$$

or

$$N_2(h) = \frac{2\sqrt{2}}{2\sqrt{2}-1} N_1(\frac{h}{2}) - \frac{1}{2\sqrt{2}-1} N_1(h)$$

Either one will do. **(8 pts)**

Answer**(4 pts)**:

$$I = N_2(h) + Ch^p + \cdots,\ p = 2$$

8. (15 pts) Find a quadrature of the form

$$\int_{-1}^{1} f(x)dx = \alpha(f(\gamma) + f(-\gamma)) + \beta f(0)$$

with largest degree of precision p, where $\alpha, \beta \in R$ and $0 < \gamma < 1$. Change the interval to $\int_{-h}^{h} f(x)dx$ and find corresponding $\alpha_h, \beta_h, \gamma_h$ in terms of h. Under the assumption (need not prove this assumption) that the error of this quadrature is of the form

$$\alpha_h(f(\gamma_h) + f(-\gamma_h)) + \beta_h f(0) = \int_{-h}^{h} f(x)dx + Kf^{(n)}(\xi)h^p,$$

find K, n and p and predict the order of convergence for the composite quadrature (i.e. what is q in $|I_h(a,b) - I(a,b)| \le Ch^q$?)

Ans:

Apply $I(1) = I_h(1)$, $I(x^2) = I_h(x^2)$ and $I(x^4) = I_h(x^4)$ to get $\alpha = \frac{5}{9}$, $\beta = \frac{8}{9}$ and $\gamma = \sqrt{\frac{3}{5}}$.**(4 pts)**

Similar procedure for $\int_{-h}^{h}$ leads to $\alpha_h = \frac{5}{9}h$, $\beta_h = \frac{8}{9}h$ and $\gamma_h = \sqrt{\frac{3}{5}}h$. The formula is also exact for all x^{2k+1} due to symmetry. So degree of precision = 5. **(3 pts)**

Therefore $n = 6$, $p = 7$ and apply $I_h(x^6)$ to get $K = -\frac{1}{15750}$. **(6 pts)**. And for the order of convergence for the composite quadrature is $q = 6$. **(2 pts)**

9. (5 pts) Use any method to find a solution of $\sqrt{1+0.9x} - \sqrt{1-0.8x} = 1.0 \times 10^{-10}$ to 15 correct digits. You need to prevent loss of accuracy. Standard methods only gives you about 5 correct digits (and no credit).

   Ans:

   Apply the following identity

$$a^2 - b^2 = (a+b)(a-b)$$

   that avoids the subtraction of two nearly identical numbers and gives

$$f(x) = \frac{1.7x}{\sqrt{1+0.9x} + \sqrt{1-0.8x}} - 10^{-10}. \ \textbf{(3 pts)}$$

   Then solve $f(x) = 0$ by any numerical method to find the solution

$$x_* \approx 1.17647058823875 \times 10^{-10}. \ \textbf{(2 pts)}$$

10. (10 pts) It is known that the unique solution to $f(x) = x + 3\sin(x) - 0.01 = 0$ is located near $x = 0$. Find a fixed point iteration that will converge for any $x_0 \in [-\frac{1}{2}, \frac{1}{2}]$. Show that your method satisfies the assumptions of a relevant Theorem, but need not prove the Theorem again. You can use the numerical values of $\sin(\frac{1}{2})$, $\cos(\frac{1}{2})$, $\exp(\frac{1}{2})$, etc. in your proof.

    Ans:

    Direct fixed point iteration with $x^{(k+1)} = g_0^{(k)}(x) = 0.01 - 3\sin(x^{(k)})$ does not converge. Instead, a proper choice of α and $g(x) = \alpha x + (1-\alpha) g_0(x)$ will result in local convergence **(2 pts)**. One could choose

$$\alpha = \frac{g_0'(\xi)}{g_0'(\xi) - 1}$$

    for some ξ near 0. Since $\xi \approx 0$, $g_0'(\xi) \approx -3$, we take $\alpha = \frac{-3}{-3-1} = \frac{3}{4}$. **(2 pts)**

    Since $g(x) = \frac{3}{4}(x - \sin x) + 0.0025$, $g'(x) = \frac{3}{4}(1 - \cos(x))$

    Therefore

$$0 < \frac{3}{4}\left(1 - \cos(\frac{1}{2})\right) \leq g'(x) \leq \frac{3}{4} \quad \text{on } [-\frac{1}{2}, \frac{1}{2}].$$

    It follows that g is an increasing function on $[-\frac{1}{2}, \frac{1}{2}]$,

$$-\frac{1}{2} < -0.012931... = g(-\frac{1}{2}) \leq g(x) \leq g(\frac{1}{2}) = 0.017931... < \frac{1}{2}$$

    (thus $g([-\frac{1}{2}, \frac{1}{2}]) \subset [-\frac{1}{2}, \frac{1}{2}]$ **(3 pts)**) and

$$|g'(x)| = \left|\frac{3}{4}(1 - \cos(x))\right| \leq \frac{3}{4} = k < 1 \ \forall \ x \in (-\frac{1}{2}, \frac{1}{2}) \ \textbf{(3 pts)}$$