Brief Answers to Quiz 06

1:20-2:10PM, Jan 08, 2021.

1. Perform the elimination part of the Gaussian elimination with **scaled partial pivoting** for the following linear system:

$$\begin{array}{rcrcrcl} x_1 & - & 5x_2 & + & x_3 & = & 7 \\ 10x_1 & & & + & 20x_3 & = & 6 \\ 5x_1 & & & - & x_3 & = & 4 \end{array}, \tag{1}$$

Give all details and the final linear system corresponding to an upper triangular matrix. Need not solve it.

Answer:

$$s_1 = 5, \quad s_2 = 20, \quad s_3 = 5.$$

Scaled linear system

$$\begin{array}{rcrcrcl} 0.2x_1 & - & x_2 & + & 0.2x_3 & = & 1.4 \\ 0.5x_1 & & & + & x_3 & = & 0.3 \\ x_1 & & & - & 0.2x_3 & = & 0.8 \end{array}, \tag{2}$$

Start with row 1 and row 3 exchange:

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & 7 \\ 10 & 0 & 20 & 6 \\ 5 & 0 & -1 & 4 \end{array}\right] \xrightarrow{E_1 \leftrightarrow E_3} \left[\begin{array}{ccc|c} 5 & 0 & -1 & 4 \\ 10 & 0 & 20 & 6 \\ 1 & -5 & 1 & 7 \end{array}\right] \textbf{ (10 pts)}$$

$$\xrightarrow{E_2 \leftarrow E_2 - 2E_1} \left[\begin{array}{ccc|c} 5 & 0 & -1 & 4 \\ 0 & 0 & 22 & -2 \\ 1 & -5 & 1 & 7 \end{array}\right] \textbf{ (6 pts)} \xrightarrow{E_3 \leftarrow E_3 - \frac{1}{5}E_1} \left[\begin{array}{ccc|c} 5 & 0 & -1 & 4 \\ 0 & 0 & 22 & -2 \\ 0 & -5 & 1.2 & 6.2 \end{array}\right] \textbf{ (6 pts)}$$

$$\xrightarrow{E_2 \leftrightarrow E_3} \left[\begin{array}{ccc|c} 5 & 0 & -1 & 4 \\ 0 & -5 & 1.2 & 6.2 \\ 0 & 0 & 22 & -2 \end{array}\right] \textbf{ (3 pts)}$$

2. Find P, L, U in the factorization $PA = LU$ for the matrix

$$A = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix} \tag{3}$$

where P is the permutation matrix corresponding to **partial pivoting**.

Answer:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_1 A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix} \quad \textbf{(5 pts)}$$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad L_1 P_1 A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \textbf{(5 pts)}$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad P_2 L_1 P_1 A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad \textbf{(2 pts)}$$

$$L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad L_2 P_2 L_1 P_1 A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \textbf{(3 pts)}$$

$$U = \begin{pmatrix} -1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad P = P_2 P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \textbf{(3 pts)}$$

$$L = P_2 (L_1)^{-1} (P_2)^{-1} (L_2)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad \textbf{(7 pts)}$$

3. Derive a system of equation corresponding to the following boundary value problem

$$\begin{aligned} &u''(x) - u(x) = f(x), \quad x \in (0,1) \\ &u(0) = 0, \ u(1) = 0 \end{aligned}$$

with uniformly spaced grids $0 = x_0 < x_1 < \cdots < x_N = 1$, $x_i - x_{i-1} = h = 1/N$, using second order finite difference method. That is, given f_i, $i = 1, 2, \cdots, N-1$, try to derive a linear system of equations to solve for u_i, $i = 1, 2, \cdots, N-1$.

Answer:

Equation at x_1: $\dfrac{0 - (2+h^2)u_1 + u_2}{h^2} = f_1,$

Equation at x_2: $\dfrac{u_1 - (2+h^2)u_2 + u_3}{h^2} = f_2,$

Equation at x_i: $\dfrac{u_{i-1} - (2+h^2)u_i + u_{i+1}}{h^2} = f_i,$

$\cdots,$

Equation at x_{N-1}: $\dfrac{u_{N-2} - (2+h^2)u_{N-1} + 0}{h^2} = f_{N-1}.$ **(25 pts)**

4. Find the Choleski decomposition for the matrix

$$A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix} \tag{4}$$

Give full details of your derivation.

Answer:
Suppose $A = GG^{\mathrm{T}}$. Since A is banded, so is G. Let

$$G = \begin{pmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ 0 & g_{32} & g_{33} \end{pmatrix}, \quad G^{\mathrm{T}} = \begin{pmatrix} g_{11} & g_{21} & 0 \\ 0 & g_{22} & g_{32} \\ 0 & 0 & g_{33} \end{pmatrix} \tag{5}$$

Equate the 11 entry: $g_{11} = 2$.

Equate the 21 entry: $g_{21} = \frac{-1}{2}$.

Equate the 22 entry: $g_{22} = \frac{\sqrt{15}}{2}$.

Equate the 32 entry: $g_{32} = \frac{-2}{\sqrt{15}}$.

Equate the 33 entry: $g_{33} = \frac{\sqrt{56}}{\sqrt{15}}$. **(5 pts for each, total 25pts)**