Brief Answers to Quiz 05

1:20-2:10PM, Dec 22, 2020.

1. Use any desingularization method to evaluate the improper integral $\int_0^1 \frac{e^x}{\sqrt{x}}\,dx$ with a composite quadrature rule of 4th order accuracy. The answer is $2.9\cdots$. Demonstrate numerically that the result is indeed 4th order accurate.

   Ans:

   $\int_0^1 \frac{e^x}{\sqrt{x}}\,dx \approx 2.92530349181436$ **(25 pts)**

   Method 1:

   See page 252-253 of the textbook.

   Method 2:

   See page 7 of NA20f_201204_Supplements_to_improper_integrals.pdf

2. Use trapezoidal rule to discretize the integral equation

$$u + \int_0^1 \exp(-|x-t|)\,u(t)\,dt = f$$

   on uniformly spaced nodes $0 = x_0 < x_1 < \cdots < x_{10} = 1$, $x_j - x_{j-1} = \frac{1}{10}$, to get a linear system of the form $Cu = f$. Construct the matrix C and evaluate $\sum_{i,j} c_{ij}$.

   Ans:

   $a = 0$, $b = 1$, $m = 10$, $h = (b-a)/m$,

   In matlab or C, let $w(j) = \begin{cases} \frac{h}{2}, & j = 1, m+1 \quad (\text{ or } j = 0, m) \\ h, & 2 \le j \le m. \quad (\text{ or } 1 \le j \le m-1.) \end{cases}$

   Then $c_{ij} = \exp(-|i-j|h) * w(j) + \delta_{ij}$. **(15 pts)**

   $\sum_{ij} c_{ij} \approx 18.98582400029024$ **(10 pts)**

3. Let A be an $M \times M$ matrix with $a_{ij} = 0$ except for $-2 \le i - j \le 2$. Suppose that $Ax = b$ can be solved using Gaussian elimination without pivoting. Find the leading order operation count (multiplication/division only) kM^p for the elimination part (ignore the backward substitution part). Give all details.

   Ans:

   There are $2M$ (to leading order) sub-diagonal elements to be eliminated. Each requires 4 multiplications/divisions (1 for calculating the ratio, 1 for each of the 2 super-diagonal elements, and 1 for the right hand side).

   Total $= 8M$ multiplications/divisions. **(25 pts)**

4. Let B be an $N^2 \times N^2$ matrix with $b_{ij} = 0$ except for $i - j = 0$, ± 1 and $\pm N$. Suppose that $Bx = c$ can be solved using Gaussian elimination without pivoting. Find the leading order operation count (multiplication/division only) kN^p for the elimination part (ignore the backward substitution part). Give all details.

Ans:

The zeros between $i-j=-N$ and $i-j=-1$ and between $i-j=1$ and $i-j=N$ are filled (become nonzero) after $2N$ steps of Gaussian elimination. The operation count is the same order as if $b_{ij}=0$ except for $|i-j| \leq N$. **(10 pts)**

In this case, there are $N^2 \times N$ (to leading order) sub-diagonal elements to be eliminated. Each requires N (to leading order) multiplications/divisions (1 for calculating the ratio, 1 for each of the N super-diagonal elements, and 1 for the right hand side).

Total $= N^4$ multiplications/divisions. **(15 pts)**