Brief Answers to Quiz 04

1:20-2:10PM, Dec 01, 2020.

1. Find the rate of convergence (the p in Ch^p) of cubic spline interpolation numerically for the function $f(x) = x^{1.5}$ on $[0, 1]$ using the steps outlined in problem 2, homework 09. The function is not smooth, so do not doubt yourself if the answer is not as good as you expected.

   Ans:

   Let $f_h(x)$ be the approximation of f using cubic spline with $x_j - x_{j-1} = h$.

   If $\max_x |f(x) - f_h(x)| \approx Ch^p$, then $p \approx \log_2 \left(\dfrac{\max_x |f(x) - f_{2h}(x)|}{\max_x |f(x) - f_h(x)|} \right)$.

   Numerical result shows that $p = 1.5$. **(25 pts)**

2. It is known that $\lim_{h\to 0^+}(1 + h)^{\frac{1}{h}} = \exp(1)$ (you need to find the rate of convergence yourself). We denote by $N_1(h) = (1 + h)^{\frac{1}{h}}$. Use Richardson extrapolation to derive the formula for $N_2(h)$. Put in all details from the beginning, then write down $\left|\dfrac{\exp(1) - N_2(0.02)}{\exp(1) - N_2(0.01)}\right|$.

   Ans:

   Use the a similar procedure as in problem 1, we find numerically that

   $$\exp(1) = N_1(h) + C_1 h + \cdots \ \textbf{(5pts)}$$

   Therefore

   $$\exp(1) = N_1(2h) + 2C_1 h + \cdots$$

   or

   $$\exp(1) = N_1(\frac{h}{2}) + C_1 \frac{h}{2} + \cdots$$

   Use either $N_1(h)$ and $N_1(2h)$ or $N_1(h)$ and $N_1(\frac{h}{2})$ to get

   $$N_2(h) = 2N_1(h) - N_1(2h), \quad \text{or } N_2(h) = 2N_1(\frac{h}{2}) - N_1(h) \ \textbf{(15pts)}$$

   Answer: $\left|\dfrac{\exp(1) - N_2(0.02)}{\exp(1) - N_2(0.01)}\right| = 3.89$ or 3.94. Either one will do. **(5 pts)**

3. Let $f_h''(x)$ be the numerical approximation of $f''(x)$ obtained from $f(x - h)$, $f(x)$ and $f(x + h)$. Write down $f_h''(x)$ and its error estimate (need not derive if you are sure about it). Find $\min_{h>0} e(h) = \min_{h>0} |f_h''(x) - f''(x)|$. Express the critical value h^* and the minimum $e(h^*)$ in terms of machine ε as $O(\varepsilon^\alpha)$ and find α's for them.

   Ans:

   $$f_h''(x) = \frac{1}{h^2}\left(f(x - h) - 2f(x) + f(x + h)\right) \ \textbf{(5 pts)}$$

$$f_h''(x) - f''(x) = \frac{h^2}{12} f^{(4)}(\xi) \text{ } \mathbf{(5\ pts)}$$

The round-off error + truncation error is bounded by

$$|fl(f_h''(x)) - f''(x)| \le |fl(f_h''(x)) - f_h''(x)| + |f_h''(x) - f''(x)| \le \frac{4\varepsilon}{h^2} + \frac{h^2}{12} M$$

where M is an upper bound of $|f^{(4)}|$. **(5 pts)**
Thus, the optimal $h^* = O(\varepsilon^{1/4})$ **(5 pts)** and $e(h^*) = O(\varepsilon^{1/2})$ **(5 pts)**.

4. Use any one of the composite quadrature methods to evaluate $\int_0^1 x^{1.5} dx$ and find the rate of convergence numerically. Again, the function is not smooth, so do not doubt yourself if the answer is not as good as you expected.

   Ans:

   Find p as in problem 1.

   Midpoint rule or Trapezoidal rule: $p = 2$.

   Simpson's rule: $p = 2.5$. **(25 pts)**