# Quiz 02

Oct 13, 2017.

1. Consider the following recursive equation $p_0 = 1, \quad p_1 = a_1, \quad p_n = \frac{10}{3}p_{n-1} - p_{n-2}$. For what values of $a_1$ is it stable in relative error? Explain.

   **Ans**: **(25 pts + extra 15 pts)**

   **(a)**: Exact solution is given by

   $$p_n^{\text{exact}} = c_1 \left(\frac{1}{3}\right)^n + c_2 3^n$$

   where $c_1 = \frac{9-3a_1}{8}$, $c_2 = \frac{3a_1-1}{8}$ ( **10 pts**)

   **(b)**: Relative error $= \left|\frac{e_n}{p_n^{\text{exact}}}\right|$. ( **8 pts**)

   Note that

   $$e_n \approx d_1 \left(\frac{1}{3}\right)^n + d_2 3^n$$

   Both $d_1$, $d_2$ are of $O(\delta)$ **(7 pts)**.

   Therefore, relative error $= \frac{O(\delta)\left(\frac{1}{3}\right)^n + O(\delta)3^n}{c_1\left(\frac{1}{3}\right)^n + c_2 3^n}$. It is stable if and only if $c_2 \neq 0$, or $a_1 \neq 1/3$. ( **extra 15 pts**)

2. The file an.txt contains a sequence stored as '$n, a_n$' at $n$th line. Find its rate of convergence. Express your answer as $O(\beta_n)$ and find $\beta_n$ explicitly. Show details.

   **Ans**: **(0 pts + extra 20 pts)**:

   Standard procedure:

   **step 1**: try semilogy and loglog plot of $|a_n - L|$ vs $n$ (where $L = \lim a_n$) to determine whether $a_n - L \approx Cn^{-p}$ or $a_n - L \approx C\alpha^{-n}$ or something else.

   **step 2**:

   Proceed to find out the constants $C, p$ or $C, \alpha$.

   **Here I forgot to put the limit** $L$ **in the problem**, therefore one can not perform step 1 easily (one could, but not straightforwardly).

   **Extra 20 pts**:

   Directly try $a_n - L \approx Cn^{-p}$ and find $p$ through

   $$p \approx \log_2 \frac{a_N - a_{2N}}{a_{2N} - a_{4N}}$$

   Different choices of $N = 30, 50, \cdots, 100, 125$ all give consistent answer $p \approx 1$ **(15 pts)**.

   **Ans**: $a_n - L = O(\frac{1}{n})$ **(5 pts)**.

3. Find a root of $x = 2\cos x$ with 10 correct decimal digits using any numerical method of your choice. Put (1): the detail formula, (2): $x_0$, $x_1$ and (3): the answer $x^*$, on the answer sheet, but need not hand in the code.

   **Ans**: **(25 pts)**

   Correct formula: **(15 pts)**, correct answer $x^* \approx 1.029866529$ **(10 pts)**

4. Show that the nonlinear equation $x = 1 + \cos(x)/2$ has a solution in $[1, 1.5]$. Let $x_0 = 1.25$, give an estimate on $N$ (need not be optimal) such that $|x_n - x^*| < 10^{-5}$ for all $n \geq N$.

   **Ans**: **(25 pts)** Existence proof: **5 pts**; Correct estimation: **(15 pts)**; Correct $N$ resulting from the estimate: **(5 pts)**.

   Possible estimate include

   **Fixed point iteration**: Let $g(x) = 1 + \frac{\cos x}{2}$. Then $|g'(x)| = |-\frac{\sin x}{2}| \leq \frac{1}{2} =: k$.

   Estimate 1:

$$|x_n - x^*| \leq k^n \max\{x_0 - a, b - x_0\} = \frac{1}{2^{n+2}} < 10^{-5}$$
$$\Rightarrow n \geq 14.60... \Rightarrow N = 15.$$

   or

   Estimate 2:

$$|x_n - x^*| \leq \frac{k^n}{1-k}|x_1 - x_0| = \frac{1}{2^{n+1}}|2\cos(1.25) - 1| < 10^{-5}$$
$$\Rightarrow n \geq 14.17... \Rightarrow N = 15.$$

   **Bisection**: Use Theorem 2.1 to estimate $|x_n - x^*|$.

   **Newton's Method**: Use $x_{n+1} - x^* = \frac{g''(\xi_n)}{2}(x_n - x^*)^2$ (section 2.4) to give an estimate.

   Any method with correct estimate will receive full 15 pts for estimate.

   Direct simulation to give $N$ will get no credits for estimate and no credits for $N$.

5. The first few iteration $(p_i, f(p_i))$, $i = 0, 1, 2, 3$ of method of false position for some equation $f(x) = 0$ is given in q2p5.txt. Find $p_4$ (4 digits will do). Also give your formula for finding $p_4$ and explain.

   **Ans**: **(25 pts)**

$$f(p_1)f(p_0) < 0 \Rightarrow a = p_0,\ b = p_1$$
$$f(p_2)f(p_1) < 0 \Rightarrow a = p_1,\ b = p_2$$
$$f(p_3)f(p_2) > 0 \Rightarrow a = p_1,\ b = p_3$$

   **(up to here = 10 pts)**

$$\Rightarrow p_4 = p_3 - f(p_3)\frac{p_3 - p_1}{f(p_3) - f(p_1)} \textbf{ (10 pts) } \approx 0.8421 \textbf{ (5 pts)}.$$