HW13

HW #1.

The theoretical prediction is 1.75 order. (Refer to the blackboard notes on improper integrals.)

- Composite Trapezoidal: Denote the numerical integral by

$$I_{h,T}(n) := \left[\int_0^1 x^{-1/4} \sin x \, dx\right]_h .$$

Check the error order numerically:

$$\log_2 \left| \frac{I_{h,T}(100) - I_{h,T}(200)}{I_{h,T}(200) - I_{h,T}(400)} \right| = 1.73...$$

```
h = 1/n;
x = 0:h:1;
f = x.^(-1/4).*sin(x);
f(1) = 0;
w = 2*ones(1,n+1);
w(1) = w(n+1) = 1;
I = w*f'*h/2;
```

- Composite Simpson's: Denote the numerical integral $I_{h,S}(n)$ similarly.

Check the error order numerically:

$$\log_2 \left| \frac{I_{h,S}(100) - I_{h,S}(200)}{I_{h,S}(200) - I_{h,S}(400)} \right| = 1.75...$$

```
h = 1/n;
x = 0:h:1;
f = x.^(-1/4).*sin(x);
f(1) = 0;
w = ones(1,n+1);
w(2:2:n  ) = 4;
w(3:2:n-1) = 2;
I = w*f'*h/3;
```

- Composite Midpoint: Denote the numerical integral $I_{h,M}(n)$ similarly.

Check the error order numerically:

$$\log_2 \left| \frac{I_{h,M}(100) - I_{h,M}(200)}{I_{h,M}(200) - I_{h,M}(400)} \right| = 1.72...$$

```
h = 1/n;
x = h/2:h:1-h/2; %to make it have almost the same data points as the other methods
f = x.^(-1/4).*sin(x);
w = ones(1,n);
I = w*f'*h;
```

- Composite Gaussian: Denote the numerical integral $I_{h,G}(n)$ similarly.

Check the error order numerically:

$$\log_2\left|\frac{I_{h,G}(100)-I_{h,G}(200)}{I_{h,G}(200)-I_{h,G}(400)}\right| = 1.74...$$

```
h = 1/(n/2);  %to make it have almost the same data points as the other methods
f = @(x) x.^(-1/4).*sin(x);
I = 0;
for i=1:n
    fi = @(t) f((h*t+(2*i-1)*h)/2)*h/2;
    I = I + fi(-1/sqrt(3)) + fi(1/sqrt(3));
end
```

HW #2.

- Composite Trapezoidal:

$$\int_0^1 x^{-1/4}\sin x\,dx = \int_0^1 x^{-1/4}(\sin x - x)\,dx + \int_0^1 x^{-1/4}\cdot x\,dx = \int_0^1 x^{-1/4}(\sin x - x)\,dx + \frac{4}{7}$$

Denote the numerical integral by

$$I_{h,T}(n) := \left[\int_0^1 x^{-1/4}(\sin x - x)\,dx\right]_h.$$

Check the error order numerically:

$$\log_2\left|\frac{I_{h,T}(100) - I_{h,T}(200)}{I_{h,T}(200) - I_{h,T}(400)}\right| = 2.00... \to \text{ 2nd order.}$$

- Composite Simpson's:

$$\int_0^1 x^{-1/4}\sin x\,dx = \int_0^1 x^{-1/4}(\sin x - x + \frac{x^3}{3!})\,dx + \int_0^1 x^{-1/4}\cdot(x - \frac{x^3}{3!})\,dx = \int_0^1 x^{-1/4}(\sin x - x + \frac{x^3}{3!})\,dx + \frac{166}{315}$$

Denote the numerical integral by

$$I_{h,S}(n) := \left[\int_0^1 x^{-1/4}(\sin x - x + \frac{x^3}{3!})\,dx\right]_h.$$

Check the error order numerically:

$$\log_2\left|\frac{I_{h,S}(100) - I_{h,S}(200)}{I_{h,S}(200) - I_{h,S}(400)}\right| = 4.00... \to \text{ 4th order.}$$

- Composite Midpoint:

$$\int_0^1 x^{-1/4}\sin x\,dx = \int_0^1 x^{-1/4}(\sin x - x)\,dx + \int_0^1 x^{-1/4}\cdot x\,dx = \int_0^1 x^{-1/4}(\sin x - x)\,dx + \frac{4}{7}$$

Denote the numerical integral by

$$I_{h,M}(n) := \left[\int_0^1 x^{-1/4}(\sin x - x)\,dx\right]_h.$$

Check the error order numerically:

$$\log_2\left|\frac{I_{h,M}(100) - I_{h,M}(200)}{I_{h,M}(200) - I_{h,M}(400)}\right| = 1.96... \to \text{ 2nd order.}$$

- Composite Gaussian:

$$\int_0^1 x^{-1/4}\sin x\,dx = \int_0^1 x^{-1/4}(\sin x - x + \frac{x^3}{3!})\,dx + \int_0^1 x^{-1/4}\cdot(x - \frac{x^3}{3!})\,dx = \int_0^1 x^{-1/4}(\sin x - x + \frac{x^3}{3!})\,dx + \frac{166}{315}$$

Denote the numerical integral by

$$I_{h,G}(n) := \left[\int_0^1 x^{-1/4} (\sin x - x + \frac{x^3}{3!})\, dx \right]_h .$$

Check the error order numerically:

$$\log_2 \left| \frac{I_{h,G}(100) - I_{h,G}(200)}{I_{h,G}(200) - I_{h,G}(400)} \right| = 4.00... \to \ \text{4th order.}$$

HW #3.

Let $t = \frac{1}{1+x}$. Then

$$\int_0^\infty \frac{1}{1+x^4}\,dx = \int_0^1 \frac{t^2}{t^4+(1-t)^4}\,dt.$$

Let $g(t) := \frac{t^2}{t^4+(1-t)^4}$. Then

$$|error| = \frac{1}{180}h^4\,|g'''(1) - g'''(0)| = \frac{1}{180}h^4\,|-120-24| < 10^{-6} \Rightarrow n > 29.9... \Rightarrow n = 30.$$

Apply Simpson's rule, then the numerical approximation is 1.11072040996602.

HW #4.

- (a) For the Trapezoidal rule $m = n = 1, x_0 = 0, x_1 = 1$ so that for $i = 0$ and 1, we have

$$u(x_i) = f(x_i) + \int_0^1 K(x_i, t)u(t)\,dt = f(x_i) + \frac{1}{2}[K(x_i, 0)u(0) + K(x_i, 1)u(1)].$$

  Substituting for x_i gives the desired equations.

The solution is

$$u(0) = \frac{2e}{1 - e^2}, \quad u(1) = \frac{2}{1 - e^2}.$$

- (b) Run the following code. Then the solution will be

$$u(0) = -0.05455598991771130,\ u(1/4) = -0.00927396706301100,\ u(2/4) = 0.15591309062303474,$$

$$u(3/4) = 0.45003541710659423,\ u(1) = 0.89051314357972045.$$

```
format long
%construct the matrix equation
n = 5;
h = 1/(n-1);
A = zeros(n,n);
x = 0:h:1;
w = 2*ones(1,n);
w(1) = w(n) = 1;
for i = 1:n
    K = -exp(-abs(x(i)-x))/2;
    A(i,:) = K.*w*h/2;
end
A = A - eye(n);
b = -x.^2;

%gaussian elimination
A = [A b'];
for i = 1:n-1
    for j = i+1:n
        m = A(j,i)/A(i,i);
        A(j,:) = A(j,:)-m*A(i,:);
    end
end

z = zeros(n,1);
z(n) = A(n,n+1)/A(n,n);
for i = n-1:-1:1
    z(i) = (A(i,n+1)-A(i,i+1:n)*z(i+1:n))/A(i,i);
end
z
```

HW #5.

Let $n = N^2$. Then the matrix is of size $n \times n$.

- Gaussian elimination: $4 \cdot 2 \cdot (n-2) + 3 \cdot 1 = 8N^2 - 13$
- Backward substitution: $3(n-2) + 2 + 1 = 3N^2 - 3$
- Leading order of the number of multiplication: $11N^2$

HW #6.

Let $n = N^2$. Then the matrix is of size $n \times n$.

- Gaussian elimination: $(N+2) \cdot N \cdot (n-N) + [(N+1)(N-1) + N(N-2) + \ldots + 3 \cdot 1] = N^4 + O(N^3)$
- Backward substitution: $(N+1)(n-N) + [N + (N-1) + \ldots + 1] = O(N^3)$
- Leading order of the number of multiplication: N^4

HW #7.

Let $n = N^2$. Then the matrix is of size $n \times n$.

- Gaussian elimination: Let k the number of step.
  - For $k = 1$, number of multiplication is $4 \cdot 2$, number of extra nonzero entries is 1.
  - For $k = 2$, number of multiplication is $5 \cdot 3$, number of extra nonzero entries is 2.
  - For $k = 3$, number of multiplication is $6 \cdot 4$, number of extra nonzero entries is 3.
  - $\vdots$
  - For $k = N - 2$, number of multiplication is $(N + 1) \cdot (N - 1)$, number of extra nonzero entries is $N - 2$.
  - For $k \geq N - 1$, the rest part becomes the same matrix as #6 of size $n' \times n'$ where $n' = n - N + 2$. Thus the leading order of the number of multiplication is $(N + 2) \cdot N \cdot (n' - N) = N^4 + O(N^3)$.
- Backward substitution: By the above reason and #6, the number of multiplication is at most $O(N^3)$.
- Leading order of the number of multiplication: N^4