HW10

HW #1.

Recall

$$f''(x_0) = \frac{1}{h^2}[f(x_0 - h) - 2f(x_0) + f(x_0 + h)] - \frac{h^2}{12} f^{(4)}(\xi), \; x_0 - h < \xi < x_0 + h.$$

Let

$$\begin{aligned} f(x_0 - h) &= \bar{f}(x_0 - h) + e(x_0 - h) \\ f(x_0) &= \bar{f}(x_0) + e(x_0) \\ f(x_0 + h) &= \bar{f}(x_0 + h) + e(x_0 + h). \end{aligned}$$

Then

$$f''(x_0) - \frac{1}{h^2}[\bar{f}(x_0 - h) - 2\bar{f}(x_0) + \bar{f}(x_0 + h)] = \frac{1}{h^2}[e(x_0 - h) - 2e(x_0) + e(x_0 + h)] - \frac{h^2}{12} f^{(4)}(\xi).$$

If $|e| \le \epsilon$ and $|f^{(4)}| \le M_4$, then

$$\left| f''(x_0) - \frac{1}{h^2}[\bar{f}(x_0 - h) - 2\bar{f}(x_0) + \bar{f}(x_0 + h)] \right| \le \frac{4\epsilon}{h^2} + \frac{h^2}{12} M_4 := e(h).$$

The critical h^* that minimizes $e(h)$ is (check!)

$$h^* = \sqrt[4]{\frac{48\epsilon}{M_4}} = O(\epsilon^{1/4}).$$

And, the corresponding error is (check!)

$$e(h^*) = O(\epsilon^{1/2}).$$

HW #2.

Recall

$$f'(x_0) = \frac{1}{12h}[f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)] + \frac{h^4}{30}f^{(5)}(\xi),\ x_0 - 2h < \xi < x_0 + 2h.$$

By the same assumptions and arguments as #1, we have

$$e(h) = \frac{3\epsilon}{2h} + \frac{h^4}{30}M_5,$$
$$h^* = O(\epsilon^{1/5}),$$

and

$$e(h^*) = O(\epsilon^{4/5}).$$

HW #3.

Consider

$$f'(x_0) = \frac{1}{2h}[f(x_0+h) - f(x_0-h)] - \frac{h^2}{6}f^{(3)}(x_0) - \frac{h^4}{120}f^{(5)}(\xi_1) \quad --(1)$$

and

$$f'(x_0) = \frac{1}{2(2h)}[f(x_0+2h) - f(x_0-2h)] - \frac{(2h)^2}{6}f^{(3)}(x_0) - \frac{(2h)^4}{120}f^{(5)}(\xi_2). \quad --(2)$$

$$(1)-(2) \Rightarrow f^{(3)}(x_0) = \frac{1}{2h^3}[f(x_0+2h) - 2f(x_0+h) + 2f(x_0-h) - f(x_0-2h)] + O(h^2)$$

where $O(h^2) \le Ch^2 |f^{(5)}(\xi)|$ for some constant C and $x_0 - 2h < \xi < x_0 + 2h$.

Consider

$$f''(x_0) = \frac{1}{h^2}[f(x_0+h) - 2f(x_0) + f(x_0-h)] - \frac{h^2}{12}f^{(4)}(x_0) - \frac{h^4}{360}f^{(6)}(\eta_1) \quad --(3)$$

and

$$f''(x_0) = \frac{1}{(2h)^2}[f(x_0+2h) - 2f(x_0) + f(x_0-2h)] - \frac{(2h)^2}{12}f^{(4)}(x_0) - \frac{(2h)^4}{360}f^{(6)}(\eta_2). \quad --(4)$$

$$(3)-(4) \Rightarrow f^{(4)}(x_0) = \frac{1}{h^4}[f(x_0+2h) - 4f(x_0+h) + 6f(x_0) - 4f(x_0-h) + f(x_0-2h)] + O(h^2)$$

where $O(h^2) \le Dh^2 |f^{(6)}(\eta)|$ for some constant D and $x_0 - 2h < \eta < x_0 + 2h$.

Textbook §4.2 #5.

$$
\begin{aligned}
N_2(h) &= N_1(\frac{h}{2}) + \frac{1}{3}[N_1(\frac{h}{2}) - N_1(h)] \\
N_2(\frac{h}{2}) &= N_1(\frac{h}{4}) + \frac{1}{3}[N_1(\frac{h}{4}) - N_1(\frac{h}{2})] \\
N_2(\frac{h}{4}) &= N_1(\frac{h}{8}) + \frac{1}{3}[N_1(\frac{h}{8}) - N_1(\frac{h}{4})] \\
N_3(h) &= N_2(\frac{h}{2}) + \frac{1}{15}[N_2(\frac{h}{2}) - N_2(h)] \\
N_3(\frac{h}{2}) &= N_2(\frac{h}{4}) + \frac{1}{15}[N_2(\frac{h}{4}) - N_2(\frac{h}{2})] \\
N_4(h) &= N_3(\frac{h}{2}) + \frac{1}{63}[N_3(\frac{h}{2}) - N_3(h)] \approx 1.999999
\end{aligned}
$$

Textbook §4.2 #10.

Let $N_2(h) = N(\frac{h}{3}) + \frac{1}{8}(N(\frac{h}{3}) - N(h))$ and $N_3(h) = N_2(\frac{h}{3}) + \frac{1}{80}(N_2(\frac{h}{3}) - N_2(h))$. Then $N_3(h)$ is an $O(h^6)$ approximation to M. (Why?)

Textbook §4.2 #12.

(a)

$$\lim_{h\to 0} \ln\left(\frac{2+h}{2-h}\right)^{\frac{1}{h}} = \lim_{h\to 0} \frac{\ln(2+h) - \ln(2-h)}{h} = \lim_{h\to 0}\left(\frac{1}{2+h} + \frac{1}{2-h}\right) = 1$$

$$\Rightarrow \lim_{h\to 0}\left(\frac{2+h}{2-h}\right)^{\frac{1}{h}} = \lim_{h\to 0} e^{\ln\left(\frac{2+h}{2-h}\right)^{\frac{1}{h}}} = e.$$

(b)

$$N(0.04) = 2.718644377221238,$$

$$N(0.02) = 2.718372444800622,$$

$$N(0.01) = 2.718304481241747.$$

(c) Let $N_2(h) = 2N(\frac{h}{2}) - N(h)$ and $N_3(h) = N_2(\frac{h}{2}) + \frac{1}{3}[N_2(\frac{h}{2}) - N_2(h)]$. Then $N_2(0.04) = 2.718100512380006$, $N_2(0.02) = 2.718236517682872$ and $N_3(0.04) = 2.718281852783827$. $N_3(0.04)$ is an $O(h^3)$ approximation satisfying $|e - N_3(0.04)| \leq 0.5 \times 10^{-7}$.

THe assumption is not correct since some coefficients k_i must be zero. For example,

$$\left|\frac{e - N_3(0.04)}{e - N_3(0.02)}\right| \approx 2^4 \neq 2^3, \quad \left|\frac{e - N(0.04)}{e - N(0.02)}\right| \approx 2^2 \neq 2^1.$$

(d)

$$N(-h) = \left(\frac{2-h}{2+h}\right)^{-\frac{1}{h}} = \left(\frac{2+h}{2-h}\right)^{\frac{1}{h}} = N(h).$$

(e) Let

$$e = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + \ldots.$$

Replacing h by $-h$ gives

$$e = N(-h) - K_1 h + K_2 h^2 - K_3 h^3 + \ldots,$$

but $N(-h) = N(h)$, so that

$$K_1 h + K_3 h^3 + \ldots = -K_1 h - K_3 h^3 - \ldots.$$

It follows that $K_1 = K_3 = K_5 = \ldots = 0$ and

$$e = N(h) + K_2 h^2 + K_4 h^4 + \dots .$$

(f) Let

$$N_2(h) = N(\frac{h}{2}) + \frac{1}{3}[N(\frac{h}{2}) - N(h)]$$

and

$$N_3(h) = N_2(\frac{h}{2}) + \frac{1}{15}[N_2(\frac{h}{2}) - N_2(h)].$$

Then

$$N_2(0.04) = 2.718281800660416,$$

$$N_2(0.02) = 2.718281826722122,$$

and

$$N_3(0.04) = 2.718281828459569$$

which is an $O(h^6)$ approximation satisfying

$$|e - N_3(0.04)| \leq 0.5 \times 10^{-12}.$$

HW #5.

Run the following code. Then you will find the solution $p_1 \approx 2$.

```
N = @(h) ((2+h)/(2-h))^(1/h);
p1 = log2((exp(1)-N(0.02))/(exp(1)-N(0.01)))
```

Textbook §4.3 #14.

By assumption,

$$\text{Trapezoidal rule} \Rightarrow f(0) + f(2) = 5$$

and

$$\text{Midpoint rule} \Rightarrow 2f(1) = 4.$$

Therefore, Simpson's rule gives

$$\int_0^2 f(x)\,dx \approx \frac{1}{3}[f(0) + 4f(1) + f(2)] = \frac{13}{3}.$$

Textbook §4.3 #19.

Let $f(x) = x^n$. Then the equality

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

holds for $n = 0, 1, 2, 3$ and doesn't hold for $n = 4$. (Why?)

Therefore, the degree of precision is 3.

Textbook §4.3 #20.

Let $f(x) = x^n$. Then the equality

$$\int_a^b f(x)\,dx = \frac{9}{4}hf(x_1) + \frac{3}{4}hf(x_2)$$

holds for $n = 0, 1, 2$ and doesn't hold for $n = 3$. (Why?)

Therefore, the degree of precision is 2.

Textbook §4.3 #22.

$$\begin{aligned} n = 0 &: c_0 + c_1 + c_2 = 2 \\ n = 1 &: c_1 + 2c_2 = 2 \\ n = 2 &: c_1 + 4c_2 = 8/3 \end{aligned}$$

$$\Rightarrow c_0 = \frac{1}{3},\ c_1 = \frac{4}{3},\ c_2 = \frac{1}{3}.$$

Textbook §4.3 #23.

$$\begin{aligned} n = 0 &: c_0 + c_1 = 1 \\ n = 1 &: c_1 x_1 = 1/2 \\ n = 2 &: c_1 x_1^2 = 1/3 \end{aligned}$$

$$\Rightarrow c_0 = \frac{1}{4},\ c_1 = \frac{3}{4},\ x_1 = \frac{2}{3}.$$

Since the formula doesn't hold for $n = 3$ (check!), the degree of precision is 2.

Textbook §4.3 #24.

$$\begin{aligned} n = 0 &: 1/2 + c_1 = 1 \\ n = 1 &: 1/2x_0 + c_1 x_1 = 1/2 \\ n = 2 &: 1/2x_0^2 + c_1 x_1^2 = 1/3 \end{aligned}$$

$$\Rightarrow c_1 = \frac{1}{2},\ x_0 = \frac{3 - \sqrt{3}}{6},\ x_1 = \frac{3 + \sqrt{3}}{6}.$$

Since the formula holds for $n = 3$ but not for $n = 4$ (check!), the degree of precision is 3.