HW9

HW #1.

Denote

$$s_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3,\ x_j \le x \le x_{j+1},\ j = 0, 1, \dots, n-1.$$

The cubic spline with not-a-knot condition gives rise to the system of equations.

$$\begin{aligned}
&s_j(x_j) = f(x_j) \Rightarrow a_j = f(x_j) \quad --(1)\\
&s_{j+1}(x_{j+1}) = s_j(x_{j+1}) \Rightarrow a_{j+1} = a_j + b_j h + c_j h^2 + d_j h^3 \quad --(2)\\
&s'_{j+1}(x_{j+1}) = s'_j(x_{j+1}) \Rightarrow b_{j+1} = b_j + 2c_j h + 3d_j h^2 \quad --(3)\\
&s''_{j+1}(x_{j+1}) = s''_j(x_{j+1}) \Rightarrow c_{j+1} = c_j + 3d_j h \quad --(4)\\
&s''' \text{ is continuous at } x_1 \text{ and } x_{n-1} \text{ (not-a-knot)} \Rightarrow d_0 = d_1,\ d_{n-2} = d_{n-1} \quad --(5)
\end{aligned}$$

$$(2), (3), (4) \Rightarrow c_{j-1} + 4c_j + c_{j+1} = \frac{3}{h^2}(a_{j-1} - 2a_j + a_{j+1}),\ j = 1, \dots, n-1 \quad --(6)$$

$$(4), (5) \Rightarrow \begin{cases} c_0 = 2c_1 - c_2 & --\ (7)\\ c_n = -c_{n-2} + 2c_{n-1} & --\ (8)\end{cases}$$

Therefore, the linear system $Ax = b$ is

$$\begin{bmatrix} 1 & -2 & 1 & & & \\ 1 & 4 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 1 & 4 & 1 \\ & & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{h^2}(a_0 - 2a_1 + a_2) \\ \vdots \\ \frac{3}{h^2}(a_{n-2} - 2a_{n-1} + a_n) \\ 0 \end{bmatrix}.$$

Replace c_0, c_n in (6) by (7), (8). Then we focus on the central $(n-1) \times (n-1)$ submatrix which derives this linear system

$$\begin{bmatrix} 6 & 0 & & & & & \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 4 & 1 & \\ & & & & 1 & 4 & 1 \\ & & & & & 0 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ \\ \\ \vdots \\ \\ \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \text{(Skip)} \\ \\ \\ \\ \end{bmatrix}.$$

Eliminate c_1, c_{n-1} of the 2nd and the last 2nd rows. Then we obtain the required linear system $\bar{A}\bar{x} = \bar{b}$

$$\begin{bmatrix} 6 & 0 & & & & & \\ 0 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 4 & 1 & \\ & & & & 1 & 4 & 0 \\ & & & & & 0 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ \\ \\ \vdots \\ \\ \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \text{(Skip)} \\ \\ \\ \\ \end{bmatrix}.$$

Clearly $\bar{A}$ is symmetric. To show that $\bar{A}$ is positive definite, consider

$$\begin{aligned} <\bar{A}\bar{x}, \bar{x}> &= 6c_1^2 + 3c_2^2 + (c_2 + c_3)^2 + 2c_3^2 + (c_3 + c_4)^2 + 2c_4^2 + \dots \\ &+ \ 2c_{n-4}^2 + (c_{n-4} + c_{n-3})^2 + 2c_{n-3}^2 + (c_{n-3} + c_{n-2})^2 + 3c_{n-2}^2 + 6c_{n-1}^2 > 0 \end{aligned}$$

for any $\bar{x} \neq 0$.

Textbook §4.1 #19.

By Eq. (4.9), the approximation is

$$\begin{aligned} f''(0.5) &= \frac{1}{0.25^2}[f(0.25) - 2f(0.5) + f(0.75)] - \frac{0.25^2}{12}f^{(4)}(\xi),\ 0.25 < \xi < 0.75 \\ &\approx \frac{1}{0.25^2}[\cos(0.25\pi) - 2\cos(0.5\pi) + \cos(0.75\pi)] \\ &= 0.00000000000000e + 00. \end{aligned}$$

The exact value is

$$f''(0.5) = -\pi^2\cos(0.5\pi) = 0.$$

The error bound is

$$\left|\frac{0.25^2}{12}f^{(4)}(\xi)\right| \le \frac{0.25^2}{12}\pi^4\cos(0.25\pi) = 0.35874...$$

The method is very accurate since the function is symmetric about $x = 0.5$ and $f^{(2k)}(0.5) = (-1)^k\pi^{2k}cos(0.5\pi) = 0,\ \forall\ k \in \mathbb{N}$.

Textbook §4.1 #24.

We have the Taylor expansions

$$\begin{aligned}
f(x_0-h) &= f(x_0) - hf'(x_0) + \frac{1}{2}h^2f''(x_0) - \frac{1}{6}h^3f'''(x_0) + \frac{1}{24}h^4f^{(4)}(x_0) + O(h^5)\\
f(x_0+h) &= f(x_0) + hf'(x_0) + \frac{1}{2}h^2f''(x_0) + \frac{1}{6}h^3f'''(x_0) + \frac{1}{24}h^4f^{(4)}(x_0) + O(h^5)\\
f(x_0+2h) &= f(x_0) + 2hf'(x_0) + 2h^2f''(x_0) + \frac{4}{3}h^3f'''(x_0) + \frac{2}{3}h^4f^{(4)}(x_0) + O(h^5)\\
f(x_0+3h) &= f(x_0) + 3hf'(x_0) + \frac{9}{2}h^2f''(x_0) + \frac{9}{2}h^3f'''(x_0) + \frac{27}{8}h^4f^{(4)}(x_0) + O(h^5).
\end{aligned}$$

Thus,

$$\begin{aligned}
&Af(x_0-h) + Bf(x_0+h) + Cf(x_0+2h) + Df(x_0+3h)\\
&= f(x_0)(A+B+C+D) + f'(x_0)h(-A+B+2C+3D) + f''(x_0)h^2\left(\frac{1}{2}A + \frac{1}{2}B + 2C + \frac{9}{2}D\right)\\
&+ f'''(x_0)h^3\left(-\frac{1}{6}A + \frac{1}{6}B + \frac{4}{3}C + \frac{9}{2}D\right) + f^{(4)}(x_0)h^4\left(\frac{1}{24}A + \frac{1}{24}B + \frac{2}{3}C + \frac{27}{8}D\right).
\end{aligned}$$

We want to eliminate the terms involving $f''(x_0), f'''(x_0)$, and $f^{(4)}(x_0)$ and have the coe!cient of $f'(x_0)$ equal 1. Thus,

$$\begin{aligned}
&-A + B + 2C + 3D = 1\\
&\frac{1}{2}A + \frac{1}{2}B + 2C + \frac{9}{2}D = 0\\
&-\frac{1}{6}A + \frac{1}{6}B + \frac{4}{3}C + \frac{9}{2}D = 0\\
&\frac{1}{24}A + \frac{1}{24}B + \frac{2}{3}C + \frac{27}{8}D = 0.
\end{aligned}$$

The solution to this linear system is

$$A = -\frac{1}{4},\ B = \frac{3}{2},\ C = -\frac{1}{2} \text{ and } D = \frac{1}{12}.$$

Thus,

$$-\frac{1}{4}f(x_0-h) + \frac{3}{2}f(x_0+h) - \frac{1}{2}f(x_0+2h) + \frac{1}{12}f(x_0+3h) = \frac{5}{6}f(x_0) + hf'(x_0) + O(h^5).$$

Solving for $f'(x_0)$ gives

$$f'(x_0) = \frac{1}{12h}[-3f(x_0-h) - 10f(x_0) + 18f(x_0+h) - 6f(x_0+2h) + f(x_0+3h)] + O(h^4).$$

Textbook §4.1 #26.

(a) Assume that the computed values $\bar{f}(x_0+h)$ and $\bar{f}(x_0)$ are related to the true values $f(x_0+h)$ and $f(x_0)$ by the formulas

$$f(x_0+h)=\bar{f}(x_0+h)+e(x_0+h)$$

and

$$f(x_0)=\bar{f}(x_0)+e(x_0).$$

The total error in the approximation becomes

$$f'(x_0)-\frac{\bar{f}(x_0+h)-\bar{f}(x_0)}{h}=\frac{e(x_0+h)-e(x_0)}{h}-\frac{h}{2}f''(\xi_0).$$

If $|e(x_0+h)|<\epsilon$, $|e(x_0)|<\epsilon$, and $|f''(\xi_0)|\le M$, then

$$\left|f'(x_0)-\frac{\bar{f}(x_0+h)-\bar{f}(x_0)}{h}\right|\le\frac{2\epsilon}{h}+\frac{hM}{2}.$$

(b) The function in Example 2 is

$$f(x)=xe^x,\ \ 1.8\le x\le 2.2.$$

We have $f'(x)=xe^x+e^x$ and $f''(x)=xe^x+2e^x$. Thus,

$$M=\max_{1.8\le x\le 2.2}|f''(x)|=f''(2.2)\approx 37.9050567.$$

The numbers in the table are given to 6 decimal places, so it is reasonable to let $\epsilon=0.0000005$. The optimal value of h is

$$h=2\sqrt{\frac{\epsilon}{M}}\approx 0.000229703.$$

Textbook §4.1 #28.

By averaging the Taylor polynomials we have (Similar to #24. Exercise!)

$$f'''(x_0) = \frac{1}{h^3}\left[-\frac{1}{2}f(x_0 - 2h) + f(x_0 - h) - f(x_0 + h) + \frac{1}{2}f(x_0 + 2h)\right] + O(h^2).$$

Textbook §4.1 #29.

Since $e'(h) = -\epsilon/h^2 + hM/3$, we have $e'(h) = 0$ if and only if $h = \sqrt[3]{3\epsilon/M}$. Also, $e'(h) < 0$ if $h < \sqrt[3]{3\epsilon/M}$ and $e'(h) > 0$ if $h > \sqrt[3]{3\epsilon/M}$, so an absolute minimum for $e(h)$ occurs at $h = \sqrt[3]{3\epsilon/M}$.