HW5

Textbook §10.1 #3ab.

a. Continuity properties can be easily shown. Moreover,

$$0 < 0.8 \leq \frac{x_1^2 + x_2^2 + 8}{10} \leq 1.25 < 1.5$$

and

$$0 < 0.8 \leq \frac{x_1 x_2^2 + x_1 + 8}{10} \leq 1.2875 < 1.5,$$

so $\mathbf{G}(x) \in D$, whenever $x \in D$.

Further,

$$\left|\frac{\partial g_1}{\partial x_1}\right| = \left|\frac{2x_1}{10}\right| \leq \frac{3}{10},$$

$$\left|\frac{\partial g_1}{\partial x_2}\right| = \left|\frac{2x_2}{10}\right| \leq \frac{3}{10},$$

$$\left|\frac{\partial g_2}{\partial x_1}\right| = \left|\frac{x_2^2 + 1}{10}\right| \leq \frac{3.25}{10},$$

and

$$\left|\frac{\partial g_2}{\partial x_2}\right| = \left|\frac{2x_1 x_2}{10}\right| \leq \frac{4.5}{10}.$$

Since

$$\left|\frac{\partial g_i(\mathbf{x})}{\partial x_j}\right| \leq 0.45 = \frac{0.9}{2},$$

for $i, j = 1, 2$, all hypothesis of Theorem 10.6 have been satisfied, and $\mathbf{G}$ has a unique fixed point in D.

b. Run the following code. Then the result will be the approximation solution is $(0.999997251768517,\ 0.999997251773238)^t$ with 0.000010 accuracy after 13 iterations.

```
p0 = [0 0]'; TOL = 10^-5; N0 = 100;

g1 = @(x) ( norm(x,2)^2 + 8 )/10;
g2 = @(x) ( x(1)*x(2)^2 + x(1) + 8 )/10;
G = @(x) [g1(x) g2(x)]';

i = 1;
p = G(p0);
while (norm(p - p0, inf) >= TOL && i <= N0)
    p0 = p;
    p = G(p0);
    i = i + 1;
end

if (i <= N0)
    fprintf('The approximation solution is (%.15f,%.15f) with %f accuracy ...
    after %d iterations.\n', p(1), p(2), TOL, i);
else
    printf('The method failed after N0 iterations, N0 = %d\n', N0);
end
```

Textbook §10.1 #4ab.

a. Let

$$\mathbf{G}(\mathbf{x}) = (g_1(\mathbf{x}),\ g_2(\mathbf{x}))^t = (x_2/\sqrt{5},\ 0.25(\sin x_1 + \cos x_2))^t$$

and

$$D = \{(x_1, x_2)|\ 0 \le x_1, x_2 \le 1\}.$$

Continuity properties can be easily shown. Moreover,

$$0 \le \frac{x_2}{\sqrt{5}} \le \frac{1}{\sqrt{5}} < 1$$

and

$$0 \le 0.25(\sin x_1 + \cos x_2) \le 0.25 \times 2 < 1,$$

so $\mathbf{G}(x) \in D$, whenever $x \in D$.

Further,

$$\left|\frac{\partial g_1}{\partial x_1}\right| = 0,$$

$$\left|\frac{\partial g_1}{\partial x_2}\right| = \frac{1}{\sqrt{5}} = 0.44721...,$$

$$\left|\frac{\partial g_2}{\partial x_1}\right| = |0.25 \cos x_1| \le 0.25,$$

and

$$\left|\frac{\partial g_2}{\partial x_2}\right| = |-0.25 \sin x_2| \le 0.25.$$

Since

$$\left|\frac{\partial g_i(\mathbf{x})}{\partial x_j}\right| \le \frac{1}{\sqrt{5}} = \frac{2/\sqrt{5}}{2},$$

for $i, j = 1, 2$, all hypothesis of Theorem 10.6 have been satisfied, and $\mathbf{G}$ has a unique fixed point in D.

b. Run the following code.

For $\mathbf{x}^{(0)} = (\frac{1}{2},\ \frac{1}{2})^t$, the result will be the approximation solution is $(0.121244020633015,\ 0.271106470503775)^t$ with 0.000010 accuracy after 10 iterations.

For $\mathbf{x}^{(0)} = (\frac{1}{4}, \frac{1}{4})^t$, the result will be the approximation solution is $(0.121240796083390, 0.271106226195433)^t$ with 0.000010 accuracy after 11 iterations.

```
% p0 = [0.5  0.5]'; TOL = 10^-5; N0 = 100;
p0 = [0.25 0.25]'; TOL = 10^-5; N0 = 100;
g1 = @(x) x(2)/sqrt(5);
g2 = @(x) 0.25*(sin(x(1))+cos(x(2)));
G = @(x) [g1(x) g2(x)]';

i = 1;
p = G(p0);
while (norm(p - p0, inf) >= TOL && i <= N0)
    p0 = p;
    p = G(p0);
    i = i + 1;
end

if (i <= N0)
    fprintf('The approximation solution is (%.15f,%.15f) with %f accuracy ...
    after %d iterations.\n', p(1), p(2), TOL, i);
else
    printf('The method failed after N0 iterations, N0 = %d\n', N0);
end
```

HW #2.

Let

$$g_1(\mathbf{x}) = -2x_2 - 0.03\sin(x_1 + x_2) + 4$$
$$g_2(\mathbf{x}) = -\frac{5}{6}x_1 - \frac{0.07}{6}\cos(x_1 - x_2) + \frac{8}{6}$$

and

$$\mathbf{G}(\mathbf{x}) = (g_1(\mathbf{x}),\ g_2(\mathbf{x}))^t$$
$$\bar{\mathbf{G}}(\mathbf{x}) = \alpha\mathbf{x} + (I - \alpha)\mathbf{G}(\mathbf{x})$$

where α is a 2×2 matrix and I is the identity matrix.

Consider

$$\mathbf{x}_{n+1} = \bar{\mathbf{G}}(\mathbf{x}_n) = \alpha\mathbf{x}_n + (I - \alpha)\mathbf{G}(\mathbf{x}_n)$$

$$\mathbf{x}_* = \bar{\mathbf{G}}(\mathbf{x}_*) = \alpha\mathbf{x}_* + (I - \alpha)\mathbf{G}(\mathbf{x}_*)$$

$$\overset{MVT}{\Rightarrow} \mathbf{x}_{n+1} - \mathbf{x}_* = \alpha(\mathbf{x}_n - \mathbf{x}_*) + (I - \alpha)\begin{bmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{z_{n,1}}) & \frac{\partial g_1}{\partial x_2}(\mathbf{z_{n,1}}) \\ \frac{\partial g_2}{\partial x_1}(\mathbf{z_{n,2}}) & \frac{\partial g_2}{\partial x_2}(\mathbf{z_{n,2}}) \end{bmatrix}(\mathbf{x}_n - \mathbf{x}_*). --(1)$$

for some $\mathbf{z}_{n,1}$, $\mathbf{z}_{n,2}$ on the straight line path connecting $\mathbf{x}_n$ and $\mathbf{x}_*$.

Since the nonlinear term is small, so the solution is near that of the linear part.

$$1x_1 + 2x_2 = 4$$
$$5x_1 + 6x_2 = 8$$

$$\Rightarrow x_1 = -2,\ x_2 = 3.$$

Therefore, we choose initial point $\mathbf{x}_0 = (-2, 3)^t$ and approximate

$$\mathbf{z}_{n,1} \approx \mathbf{z}_{n,2} \approx \mathbf{x}_0. --(2)$$

Combine (1) and (2), then

$$\mathbf{x}_{n+1} - \mathbf{x}_* \approx \alpha(\mathbf{x}_n - \mathbf{x}_*) + (I - \alpha)D\mathbf{G}(\mathbf{x}_0)(\mathbf{x}_n - \mathbf{x}_*) = (\alpha + (I - \alpha)D\mathbf{G}(\mathbf{x}_0))(\mathbf{x}_n - \mathbf{x}_*).$$

To accelerate the convergence, we choose α such that

$$\alpha + (I - \alpha)D\mathbf{G}(\mathbf{x}_0) = 0$$

$$\Rightarrow \alpha = (D\mathbf{G}(\mathbf{x}_0) - I)^{-1}D\mathbf{G}(\mathbf{x}_0).$$

Run the following code. Then the result will be the approximation solution is $(-1.970058443154652,\ 2.972388308604963)^t$ with 0.000010 accuracy after 3 iterations.

```
p0 = [-2 3]'; TOL = 10^-5 ; N0 = 100;

dg1dx1 = @(x) -0.03*cos(sum(x));
dg1dx2 = @(x) -2 - 0.03*cos(sum(x));
dg2dx1 = @(x) -5/6 + 0.07*sin(-diff(x))/6;
dg2dx2 = @(x) -0.07*sin(-diff(x))/6;
DGp0 = [dg1dx1(p0) dg1dx2(p0); dg2dx1(p0) dg2dx2(p0)];
al = (DGp0 - eye(2))\DGp0;

g1 = @(x) -2*x(2) - 0.03*sin(sum(x)) + 4;
g2 = @(x) -5*x(1)/6 - 0.07*cos(-diff(x))/6 + 8/6;
G = @(x) al*x + (eye(2)-al)*[g1(x) g2(x)]';

i = 1;
p = G(p0);
while (norm(p - p0, inf) >= TOL && i <= N0)
    p0 = p;
    p = G(p0);
    i = i + 1;
end

if (i <= N0)
    fprintf('The approximation solution is (%.15f,%.15f) with %f accuracy ...
    after %d iterations.\n', p(1), p(2), TOL, i);
else
    printf('The method failed after N0 iterations, N0 = %d\n', N0);
end
```

Textbook §10.2 #7ab.

Run the following code.

a. The approximation solution is $(0.500000000000000,\ 0.866025403784439)^t$ with 0.000001 accuracy after 5 iterations.

b. The approximation solution is $(1.772453850905516,\ 1.772453850905516)^t$ with 0.000001 accuracy after 6 iterations.

```
%#7a
% p = [1 1]'; TOL = 10^-6; N0 = 100;
% f1 = @(x) 3*x(1)^2-x(2)^2;
% f2 = @(x) 3*x(1)*x(2)^2-x(1)^3-1;
% df1dx1 = @(x) 6*x(1);
% df1dx2 = @(x) -2*x(2);
% df2dx1 = @(x) 3*x(2)^2-3*x(1)^2;
% df2dx2 = @(x) 6*x(1)*x(2);

%#7b
p = [2 2]'; TOL = 10^-6; N0 = 100;
f1 = @(x) log(norm(x,2)^2)-sin(x(1)*x(2))-log(2)-log(pi);
f2 = @(x) exp(-diff(x))+cos(x(1)*x(2));
df1dx1 = @(x)  2*x(1)/(norm(x,2)^2)-x(2)*cos(x(1)*x(2));
df1dx2 = @(x)  2*x(2)/(norm(x,2)^2)-x(1)*cos(x(1)*x(2));
df2dx1 = @(x)  exp(-diff(x))-x(2)*sin(x(1)*x(2));
df2dx2 = @(x) -exp(-diff(x))-x(1)*sin(x(1)*x(2));

i = 1;
while (i <= N0)
    Fp = [f1(p) f2(p)]';
    Jp = [df1dx1(p) df1dx2(p); df2dx1(p) df2dx2(p)];
    y = -Jp\Fp;
    p = p + y;
    if (norm(y,inf) < TOL)
        fprintf('The approximation solution is (%.15f,%.15f) with %f accuracy ...
        after %d iterations.\n', p(1), p(2), TOL, i);
        return;
    end
    i = i + 1;
end
printf('The method failed after N0 iterations, N0 = %d\n', N0);
return;
```

Textbook §10.2 #14.

Since $f_j(x_1, \ldots, x_n) = a_{j1}x_1 + \ldots + a_{jn}x_n - b_j$, we have $\frac{\partial f_j}{\partial x_i} = a_{ji}$.

Hence,

$$J(\mathbf{x}) = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} = A.$$

Further,

$$\mathbf{F}(\mathbf{x}^{(0)}) = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = J(\mathbf{x}^{(0)})\mathbf{x}^{(0)} - \mathbf{b}.$$

Thus, given $\mathbf{x}^{(0)}$, we have

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - J(\mathbf{x}^{(0)})^{-1}(J(\mathbf{x}^{(0)})\mathbf{x}^{(0)} - \mathbf{b}) = A^{-1}\mathbf{b}.$$

So given any $\mathbf{x}^{(0)}$, the solution to the linear system is $\mathbf{x}^{(1)}$.