HW4

Textbook §2.4 #7a.

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^k}}{\frac{1}{n^k}} = 1.$$

Textbook §2.4 #8.

(a)

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \lim_{n\to\infty} 1 = 1.$$

(b)

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{(10^{-n^k})^2} = \lim_{n\to\infty} 10^{n^k\left[2-\left(\frac{n+1}{n}\right)^k\right]} = \infty.$$

Textbook §2.4 #9.

(a)

$$p_n = 10^{-3^n} \Rightarrow \lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^3} = \lim_{n\to\infty} \frac{10^{-3^{n+1}}}{10^{-3^{n+1}}} = 1.$$

(b)

$$p_n = 10^{-\alpha^n} \Rightarrow \lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^\alpha} = \lim_{n\to\infty} \frac{10^{-\alpha^{n+1}}}{10^{-\alpha^{n+1}}} = 1.$$

Textbook §2.4 #10.

Suppose that for $x \neq p$

$$f(x) = (x-p)^m q(x)$$

where $\lim_{x \to p} q(x) \neq 0$. Then

$$g(x) = x - \frac{m(x-p)q(x)}{mq(x) + (x-p)q'(x)}.$$

By direct computations (exercise!), $g'(p) = 0$.

Textbook §2.4 #12.

"⇒" Suppose that for $x \neq p$

$$f(x) = (x-p)^m q(x)$$

where $\lim_{x\to p} q(x) \neq 0$. Then clearly

$$f(p) = \lim_{x\to p} f(x) = 0.$$

For $1 \leq k \leq m$,

$$\begin{aligned} f^{(k)}(x) = & \sum_{j=0}^{k} \binom{k}{j} \frac{d^j (x-p)^m}{dx^j} q^{(k-j)}(x) \\ & = (x-p)^m q^{(k)}(x) + \sum_{j=1}^{k} \binom{k}{j} m(m-1)\ldots(m-j+1)(x-p)^{m-j} q^{(k-j)}(x). \end{aligned}$$

Thus for $1 \leq k \leq m-1$,

$$f^{(k)}(p) = \lim_{x\to p} f^{(k)}(x) = 0.$$

And for $k = m$,

$$f^{(m)}(p) = \lim_{x\to p} f^{(m)}(x) = m! \lim_{x\to p} q(x) \neq 0.$$

"⇐" Suppose that

$$f(p) = f'(p) = \ldots = f^{(m-1)}(p) = 0 \text{ and } f^{(m)}(p) \neq 0.$$

Consider the $(m-1)$th Taylor polynomial of f at p

$$\begin{aligned} f(x) &= f(p) + f'(p)(x-p) + \ldots + \frac{f^{(m-1)}(p)(x-p)^{m-1}}{(m-1)!} + \frac{f^{(m)}(\xi(x))(x-p)^m}{m!} \\ &= (x-p)^m \frac{f^{(m)}(\xi(x))}{m!} \end{aligned}$$

where $\xi(x)$ is between x and p. Let

$$q(x) = \frac{f^{(m)}(\xi(x))}{m!}.$$

Then $f(x) = (x-p)^m q(x)$ and $\lim_{x\to p} q(x) = \frac{f^{(m)}(p)}{m!} \neq 0$.

Textbook §2.4 #13.

We check the cubic convergence.

$$
\begin{aligned}
g(p_n) &= g(p) + g'(p)(p_n - p) + \frac{g''(p)}{2!}(p_n - p)^2 + \frac{g'''(\xi_n)}{3!}(p_n - p)^3 \\
\Rightarrow |p_{n+1} - p| &= |g(p_n) - p| = \left| \frac{g'''(\xi_n)}{3!}(p_n - p)^3 \right| \\
\Rightarrow \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^3} &= \lim_{n\to\infty} \frac{|g'''(\xi_n)|}{3!} = \frac{|g'''(p)|}{3!}
\end{aligned}
$$

The "Expand the analysis in Example 1..." part is a mistake due to the revision in the 10th edition. In the previous editions, Example 1 is now called Illustration on page 79. Please ignore this part.

Remark.

Let $f(x_*) = 0$ and $f(x_n) = y_n$. Consider

$$
x_* = f^{-1}(0) \approx f^{-1}(y) - (f^{-1})'(y)y + \frac{(f^{-1})''(y)}{2}y^2 =: g(y)
$$

$$
\begin{aligned}
\Rightarrow x_n &= g(y_{n-1}) = f^{-1}(y_{n-1}) - (f^{-1})'(y_{n-1})y_{n-1} + \frac{(f^{-1})''(y_{n-1})}{2}y_{n-1}^2 \\
&= x_{n-1} - \frac{1}{f'(x_{n-1})}f(x_{n-1}) - \frac{f''(x_{n-1})}{2f'(x_{n-1})^3}f(x_{n-1})^2 \\
&= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} - \frac{f''(x_{n-1})}{2f'(x_{n-1})}\left[\frac{f(x_{n-1})}{f'(x_{n-1})}\right]^2.
\end{aligned}
$$

Textbook §2.4 #14.

Let $e_n = p_n - p$. If

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda > 0,$$

then for sufficiently large values of n, $|e_{n+1}| \approx \lambda |e_n|^\alpha$

$$\Rightarrow |e_{n-1}| \approx \lambda^{-1/\alpha} |e_n|^{1/\alpha}. \quad --(1)$$

Using the hypothesis gives

$$\lambda |e_n|^\alpha \approx |e_{n+1}| \approx C|e_n||e_{n-1}|. \quad --(2)$$

By (1) and (2),

$$\begin{aligned} &\lambda |e_n|^\alpha \approx C|e_n|\lambda^{-1/\alpha}|e_n|^{1/\alpha} \\ \Rightarrow\ &|e_n|^\alpha \approx C\lambda^{-1/\alpha-1}|e_n|^{1/\alpha+1} \\ \Rightarrow\ &\alpha = \frac{1}{\alpha} + 1 \\ \Rightarrow\ &\alpha = \frac{1+\sqrt{5}}{2}. \end{aligned}$$

Textbook §2.5 #12a.

Run the following code, then the result will be the approximation solution is 2.554195952837043 with 0.000010 accuracy after 3 iterations.

```
%p0 = 2; TOL = 10^(-5); N0 = 100;
p0 = 2.5; TOL = 10^(-5); N0 = 100;
g = @(x) 2 + sin(x);
i = 1;
while (i <= N0)
    p1 = g(p0);
    p2 = g(p1);
    p = p0 - (p1-p0)^2/(p2-2*p1+p0);
    if (abs(p-p0) < TOL)
        fprintf('The approximation solution is %.15f with %f accuracy ...
        after %d iterations.\n', p, TOL, i);
        return;
    end
    i += 1;
    p0 = p;
end
printf('The method failed after N0 iterations, N0 = %d\n', N0);
```

Textbook §2.5 #14.

(a) If $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = \lambda > 0$, then

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^\alpha} \cdot |p_n-p|^{\alpha-1} = \lambda \cdot 0 = 0.$$

(b) For superlinear convergence,

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)^{n+1}}}{\frac{1}{n^n}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} = \frac{1}{e} \cdot 0 = 0.$$

Let $\alpha = 1 + r$ for some $r > 0$. Then

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)^{n+1}}}{\left(\frac{1}{n^n}\right)^\alpha} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \frac{n^{nr}}{n+1} = \infty.$$

Textbook §2.5 #16.

Let $\delta_n = \frac{p_{n+1}-p}{p_n-p} - \lambda$, then $\lim_{n\to\infty} \delta_n = 0$. By direct computations (exercise!),

$$\frac{\hat{p_n} - p}{p_n - p} = \frac{\lambda(\delta_n + \delta_{n+1}) - 2\delta_n + \delta_n\delta_{n+1} - 2\delta_n(\lambda - 1) - \delta_n^2}{(\lambda - 1)^2 + \lambda(\delta_n + \delta_{n+1}) - 2\delta_n + \delta_n\delta_{n+1}} \to 0 \text{ as } n \to \infty.$$

HW #3.

(1) Claim: $\lim_{n\to\infty} \frac{p_{n+1}-p}{p_n-p} = 1$.

$$\lim_{n\to\infty} \frac{\cos\frac{1}{n+1} - 1}{\cos\frac{1}{n} - 1} = \lim_{n\to\infty} \frac{\sin\frac{1}{n+1} \cdot \frac{1}{(n+1)^2}}{\sin\frac{1}{n} \cdot \frac{1}{n^2}} = 1.$$

(2) Claim: $\lim_{n\to\infty} \frac{\hat{p_n}-p}{p_{n+2}-p} = \frac{1}{3}$.

Use $p_n = \cos\frac{1}{n} \approx 1 - \frac{1}{2n^2}$. Then by direct computations,

$$\frac{\hat{p_n} - p}{p_{n+2} - p} = \frac{p_n - p - \frac{(p_{n+1}-p_n)^2}{(p_{n+2}-p_{n+1})-(p_{n+1}-p_n)}}{p_{n+2} - p} \approx \frac{-\frac{1}{2n^2} - \frac{\left(-\frac{1}{2n^2}\right)'^2}{\left(-\frac{1}{2n^2}\right)''}}{-\frac{1}{2(n+2)^2}} = \frac{\frac{1}{6n^2}}{\frac{1}{2(n+2)^2}} \to \frac{1}{3}$$

as $n \to \infty$.

(3) Claim: $\lim_{n\to\infty} \frac{\hat{p_{n+1}}-p}{\hat{p_n}-p} = 1$.

$$\frac{\hat{p_{n+1}} - p}{\hat{p_n} - p} = \frac{\hat{p_{n+1}} - p}{p_{n+3} - p} \frac{p_{n+3} - p}{p_{n+2} - p} \frac{p_{n+2} - p}{\hat{p_n} - p} \to \frac{1}{3} \cdot 1 \cdot 3 = 1.$$

as $n \to \infty$.

(4) Verify the result of (2) numerically.

```
n = 100;
p = @(x) cos(1/x);
pa = @(x) p(x) - (p(x+1)-p(x))^2 / (p(x+2)-2*p(x+1)+p(x));

e = @(x) p(x) - 1;
ea = @(x) pa(x) - 1;

for i = 1:n
        fprintf('n = %3.0f %16.8f\n', i, ea(i) / e(i+2));
end
```

For n = 100, the result will be 0.33995921.